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A review of SIF evaluation and modelling of singularities in BEM

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Abstract Various boundary element method (BEM) based approaches to solve crack problems are discussed. The displacement method, J-integral method and the modified crack closure integral (MCCI) method for the evaluation of the stress intensity factors (SIFs) are reviewed. Effects of partial and total modelling of singularities on the accuracy of the results have been presented. Elements capable of partial and total modelling of the wellknown square root singularities, variable order singularities, neighbouring variable order singularities, etc., are also reviewed. Case studies are included to illustrate the effectiveness of the various methods of calculation of the SIFs and the performance of the special elements.

List of symbols

a	crack length
c, d	singularity parameter
c_n	coefficients used in MCCI calculation
E	elastic modulus
G_{I}, G_{II}	strain energy release rate in mode I, mode II
h, L, W	domain geometric dimensions
K_{I}, K_{II}	stress intensity factors
K_{T}	amplitude of thermal singularity
l	crack tip element size/length of microcrack
N_i, M_i	shape functions
p, q	components of crack edge loading normal and parallel to crack
r	distance from crack tip
s, t	components of traction parallel and normal to crack
u, v	components of displacement parallel and normal to crack
W_{I}, W_{II}	crack closure work
x, y	Cartesian co-ordinates
Y	SIF correction factor
Δa	kink length
α	coefficient of thermal expansion
δ	distance from main crack tip to that of neighbouring microcrack
θ	crack orientation with x-axis
μ	shear modulus
ν	Poisson's ratio
ξ	natural coordinate
ϕ	potential/temperature
λ	potential/temperature gradient
Γ	boundary of a domain

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1 Introduction

The stress intensity factor (SIF) plays a vital role in the application of the principles of linear elastic fracture mechanics (LEFM) to practice. The determination of SIF for real-life components with crack is therefore very important. The SIFs can be determined through analytical, numerical and experimental methods (Rooke and Cartwright, 1976; Murakami, 1987). The analytical methods are mostly suitable for idealised geometries, loading and boundary conditions. An up-to date and exhaustive discussion of computational fracture mechanics methodologies is available in Atluri (1997). The numerical methods, e.g. the finite element method (FEM) and the boundary element method (BEM), have proved very useful for practical geometries. Several investigators, e.g. Watwood (1969), Chan et al. (1970), Parks (1974), Barsoum (1976), Tracey and Cook (1977), etc., have employed FEM to evaluate the SIFs. Jawson (1963), Symm (1963), Rizzo (1967), Cruse (1969), etc., have applied BEM for fracture mechanics applications. Recently, Cruse (1997) has reviewed the 25 years of developments of fracture mechanics analysis using BEM.

Analysing a mixed mode crack problem, where both the crack edges are modelled in a single domain, leads to the degeneracy of the boundary element (BE) equations. This can be avoided adapting either the subregion technique (Blandford et al., 1981) or the dual boundary element method (DBEM) (Portela and Aliabadi, 1992a). A crack can be subjected to remote mechanical loading, crack edge loading (e.g. fluid pressure), thermal loading, etc. While the mechanical loading can be taken care of in a routine manner in the BEM, the thermal loading requires special attention (Rizzo and Shippy, 1977).

It is wellknown that the SIFs can be obtained by the displacement method and energy method in a numerical technique. One of the versatile and accurate energy based method is the modified crack closure integral (MCCI) procedure. The MCCI technique is first implemented in the FEM by Rybicki and Kanninen (1977). Later many investigators have employed this and shown the effectiveness of this procedure in an accurate calculation of the

SIFs. Recently this method has also been adapted in the BEM. The various methods of evaluation of the SIFs in general and the MCCI in particular have been reviewed in the present paper.

In the BEM an analysis of crack involves modelling of the crack tip singularities. While in the displacement finite element formulation, the modelling of strain singularity automatically ensures the stress singularity, in the BEM the strain and stress (or traction) singularities are to be modelled separately. From the standpoint of practical applications of the BEM it is important to know the influences of partial (i.e. strain singularity) and total (i.e. both strain and stress singularities) modelling on the accuracy of computation of the SIFs. The partial and total modelling of singularities have been discussed in Sect. 4.

The most common singularity at the crack tip is the square root singularity. There are many applications involving normal crack terminating at a bimaterial interface, kinked crack, etc., where the order of singularity can be variable. In the case of a kinked crack the order of singularity varies with the knee angle (Williams, 1952); in the case of the normal crack in a bimaterial the order depends on the material combinations (Cook and Erdogan, 1972). A large number of elements are available in the FEM (e.g., Tracey and Cook, 1977) to model the field around such a crack tip. Relatively special boundary elements which can help to model the field either partially or fully is lacking. The simulation of variable order singularities is reviewed in the Sect. 4.

Neighbouring singularities come up in a domain if there is a small crack, or a large crack surrounded by a number of micro- or small cracks or a zigzag crack. The most obvious approach to tackle such cases is to employ a large number of refined elements in the neighbourhood of the crack tips. It has been shown in the FEM that the most challenging and attractive way (e.g. Maiti, 1992a) to handle the situations is to employ the multipoint singularity elements. The related investigations for simulation of multipoint singularities to solve crack-crack interaction problems are also reviewed.

Some of the advances in the BEM has benefited from the earlier advances in the FEM. So a frequent references to the latter is in order.

2
BEM applications to fracture mechanics

Some of the typical problems that come up in this application are briefly discussed in the following. A very useful discussion is also provided by Cruse (1997).

2.1
Degeneracy of boundary integral equation (BIE)

The BEM is first applied to problems of fracture mechanics by Cruse (1969). The early results pertain to problems with limited bending. It is reported that, through a piecewise constant modelling of the boundary displacements, it is possible to capture the essential nature of the crack opening behaviour but there is an error which can be as high as 10–20% (Cruse, 1997).

The BIE formulation for a fracture mechanics problem concerns with a geometry for which there are two surfaces in close proximity and across which the displacements are discontinuous. The BIE is found to degenerate for such situations (Cruse, 1972). In the case of modelling of mixed mode problems by the BEM a close proximity of nodes on the two crack edges, the displacement BIE degenerates to a singular form. The effective numerical strategy that is suggested to overcome this limitation uses multidomain or subregion modelling. The first use of a multidomain BEM is reported by Lachat and Watson (1976). In this approach the domain is broken into number of subdomains by extending the crack edges/surfaces through the body artificially (Fig. 1). The first true multidomain fracture mechanics analysis for non-symmetric bodies has been reported by Blandford et al. (1981).

In the multidomain or subregion method, the two edges of a crack are associated with the two domains (Fig. 1b). The domains are connected through an interface. Solution of this type of a problem requires a special handling of the nodes on the common interface as they do not have any prescribed displacement (or temperature in a heat conduction problem) or traction (or temperature gradient). Two additional conditions are imposed at each of these nodes to arrive at a solution. These include a condition of the continuity of displacements (or temperatures) and

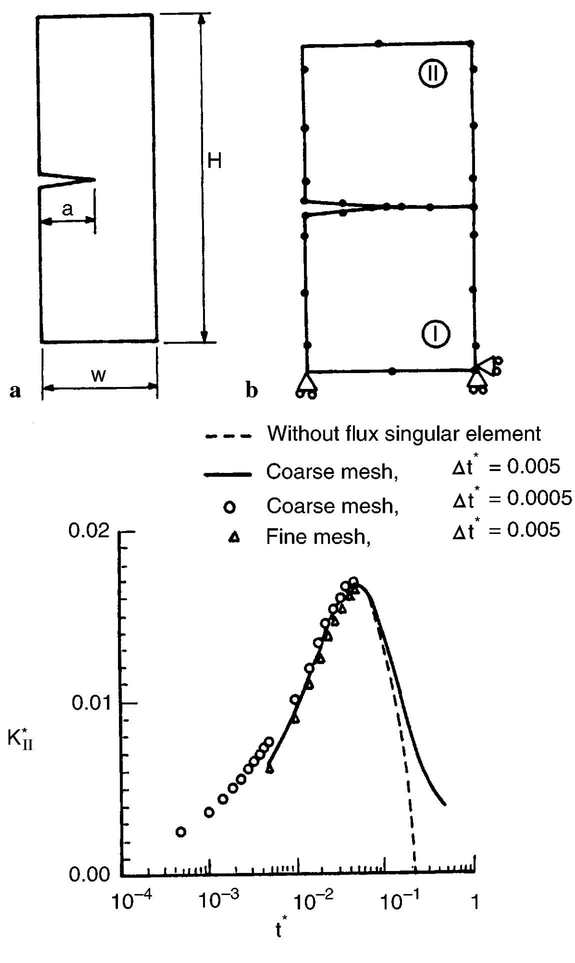


Fig. 1a–c. Edge crack under thermal load. **a** Geometry. **b** Mesh and **c** Variation of SIF due to thermal shock with time (Katsareas and Anifantis, 1995)

balance of tractions (or temperature gradients). Each of the subdomain is modelled separately by the BEM. The two systems of equations are then combined using the constraint conditions at the common nodes. Katsareas and Anifantis (1995) have solved an edge crack (Fig. 1a) under mode II thermal loading using the subregion technique. The plate is divided into two domains I and II connected through a common interface (Fig. 1b). The two crack edges are considered in separate domains.

The subregion technique can be adapted to solve problems of stick contact, Coulomb friction slip contact, interference contact, etc. As has been pointed out by Cruse (1997) the real shortcoming of this method is the need to create artificial edges/surfaces extending from the original crack tip to the physical boundaries. Such a discretization of the interior of the body is a problem when one wants to employ the elastic fracture mechanics solution to predict the path of crack growth in a way that is not dictated by an artificial boundary. A large number of investigators, e.g. Blandford et al. (1981), Martinez and Dominguez (1984), Karami and Fenner (1986), Ang (1986), Katsareas and Anifantis (1995), Katsareas et al. (1998), etc., have employed this technique to analyse mixed mode crack problems.

2.2
Thermoelastic problem

The thermoelastic problems are concerned with the determination of stresses caused by a temperature gradient and/or external restraint imposed on the free expansion of a body. The calculation of temperature and temperature gradient is a prerequisite to study a thermoelastic problem. This temperature and temperature gradient are used as an input for the computation of thermoelastic stresses. The thermal loading is treated as a body force term. The body force term is a volume/area integral which normally requires the discretisation of the body into volume/area elements. Rizzo and Shippy (1977) introduced a procedure where the interior discretisation is not required and the BIE can be expressed as an area/boundary integral, i.e. the dimensionality can still be reduced by one. Evaluation of SIFs due to thermal loading is available in Lee and Cho (1990), Raveendra and Banerjee (1992), Raveendra et al. (1993), Sladek and Sladek (1993, 1997), Prasad et al. (1994, 1996), Katsareas and Anifantis (1995), Katsareas et al. (1998), etc.

2.3
Unique integral formulations to overcome BIE degeneracy

There are four ways of overcoming the degeneracy associated with crack problems in the BIE. These are: multi-domain formulation, dual boundary element method (Portela and Aliabadi, 1992a; Guimaraes and Telles, 1994, etc.), displacement discontinuity method (Sladek and Sladek, 1986; Polch et al., 1987; Cruse, 1988; Balas et al., 1989; Richardson and Cruse, 1998; etc.) and Green's function technique (Synder and Cruse, 1975; Rudolphi and Koo, 1985; Melnikov, 1995; etc.).

Portela and Aliabadi (1992a), Portela et al. (1992b), Guimaraes and Telles (1994), Selcuk et al. (1994), Gray and Paulin (1997), etc., have shown that the mixed mode crack problems can be solved using hypersingular boundary integral equations. This method is also known as the Dual Boundary Element Method (DBEM). This does not require any subdivisioning of the geometry. In the DBEM, the singularity in the final system of equations is avoided by using two different equations for the boundary nodes on the opposite crack edges. The displacement equation is applied to one crack edge/surface and traction equation to the other crack edge/surface. This method offers some advantages over the subregion method in the study of crack extensions. However the formulation is complex and numerical integration is complicated. Raveendra et al. (1993) have commented that in the DBEM the integral equation becomes hypersingular and also the formulation requires smoothness of the shape functions which can be difficult in three-dimensional problems involving irregular crack shapes.

In the displacement discontinuity (DD) method the magnitude of discontinuity over the crack segment is taken as constant (Crouch and Starfield, 1983). A set of such discontinuities is applied over the crack. This approach does not work as well when there is significant bending. A variety of regularisation methods have been developed to eliminate the hyper- and Cauchy-singular terms in the Somigliana stress identity to weakly singular and easily integrable equations, e.g. Krishnasamy et al. (1992). In most cases, the requirements of continuity of derivative of the crack dislocations have been imposed. Though these are non-standard requirements, they result in a reduced computation and modelling simplicity for cracks. A comprehensive discussion on displacement discontinuity method, hyper- and Cauchy-singularity issues are available in Cruse (1997).

Another unique and powerful BIE formulation for fracture mechanics problems is the exact representation of the crack surface boundary conditions through the Green's functions. Analytical Green's function methods are generally considered to be limited to two dimensional problems. The earliest numerical application of such a special Green's function is given by Snyder and Cruse (1975). The Green's function for the two dimensional crack problem amounts to combining the usual Kelvin's fundamental solution with additional terms in the kernels that result in zero stresses on the crack due to the Kelvin point load. The Green's function based formulation confirms the inverse square root behaviour of the stresses and strains near a crack tip. Based on the analytical results it is possible to take the limiting form of the strains times the square root of the distance from the crack tip. The analytical limit is proportional to the crack tip SIFs. This results in a path independent integral for the mixed mode SIFs. The use of Green's functions is discussed by several investigators, e.g. Synder and Cruse (1975), Rudolphi (1985), Cruse and Raveendra (1988), Li and Chudnovsky (1994), Melnikov (1995), Chen and Hasebe (1997), etc.

2.4
Elastodynamic analysis

In elastodynamics, a knowledge of time dependent asymptotic stress and displacement fields near the crack tip and the associated parameters (e.g. K_{Id}, $\hat{J}$, J'_k, etc.) are important. Such information can help in understanding the fast fracture of solids. As an example, Nishioka and Atluri (1983) have proposed a method to evaluate J_k' using the complex potential method. Simple formulae for determining SIFs from the complex potentials are also given. The dynamic energy release rates are also calculated through the crack closure integral.

Solutions in elastodynamics using BEM are usually obtained by either the time domain method or transform method (Laplace or Fourier) or the dual reciprocity method. The time domain method is used by Nishimura et al. (1988) to solve both two and three dimensional crack problems. Dominguez and Gallego (1992) developed the subregion formulation for 2-D crack problems. Sladek and Sladek (1986) used the Laplace transform method. Chirino and Dominguez (1989) have adapted the subregion analysis and the Fourier transform method. Fedelinski et al. (1995) have developed a general formulation for modelling elastodynamic crack problems in a single domain by the DBEM. The DBEM is further extended by Wen et al. (1998) to address 3-D problems subjected to dynamic loading.

3
Determination of SIFs

The BEM has found an extensive application for an evaluation of the SIFs. The notable contributions towards the evaluation of the SIFs through the BEM are due to Cruse and Buren (1971), Cruse (1972, 1978), Cruse and Wilson (1978), Tan and Fenner (1978, 1979) in the seventies. Blandford et al. (1981) have proposed a multidomain formulation for the mixed mode problems. The BEM has also been applied to axisymmetric crack problems (Bakr and Fenner, 1985), fatigue crack growth (Gerstle et al., 1988), three dimensional fracture problems (Perucchio and Ingraffea, 1985), analysis of mixed mode crack problems using contact mechanics (Liu and Tan, 1992), surface cracks (Zeng et al., 1993), dissimilar materials with interface cracks (Yuuki and Xu, 1994), orthotropic delamination specimens (Ang et al., 1996), etc.

There are various methods, e.g. displacement method, energy based methods, etc., for the evaluation of SIFs. The commonly employed methods are discussed in the following.

3.1
Displacement and stress methods

This method is based on Irwin's classical solution for the displacements in the vicinity of the crack tip. Watwood (1969) and Chan et al. (1970) first introduced it in the FEM.

$$K_\mathrm{I} = v f(E, \nu)/\sqrt{r} \qquad (1)$$

where v denotes the normal displacement of the crack face and $f(E, \nu)$ is a function of elastic modulus E and Poisson ratio ν. This expression is valid as $r \to 0$. A similar expression can be written for K_II involving crack sliding displacements u. It is observed that the accuracy in this method is dependent on the mesh refinement near the crack tip. A set of SIFs can also be determined considering a number of nodes along a radial line emanating from the crack tip. Employing these nodal values the SIF at $r = 0$ can be obtained through a graphical extrapolation. This is the displacement extrapolation method (Chan et al., 1970). The displacement method has been straightway adapted in the BEM. This method is simple, versatile and more widely used.

The SIF can also be obtained by the BEM based stresses instead of the displacement.

$$K_\mathrm{I} = (\sigma_n \sqrt{r}) f \qquad (2)$$

where f is a constant and σ_n denotes the stress normal to crack in the region ahead of the crack tip. The extrapolation scheme too can be employed in this case to evaluate the SIF.

Blandford et al. (1981) and Martinez and Dominguez (1984) have shown that the accuracy of the results through the displacement method is good though it depends on the mesh refinement near the crack tip. Katsareas and Anifantis (1995) have employed the displacement and traction formulae to evaluate the SIFs due to a thermal shock. A time domain BEM is employed for a problem subjected to a thermal shock. An edge cracked finite plate (Fig. 1) under thermal shock is considered by Katsareas and Anifantis (1995). Initially the plate is at zero temperature. Suddenly the top and bottom edge temperatures are brought to +1°C and −1°C respectively. The subregion method is adapted to examine the case. The dimensionless SIFs for plane strain conditions ($K_\mathrm{II}^* = K_\mathrm{II}(1-\nu)/[\alpha E(\theta_T - \theta_B)]$) are shown in Fig. 1c.

3.2
Stiffness derivative procedure

The strain energy release rate G, is the rate of change of strain energy for an incremental change in crack length ∂a

$$G = \frac{\partial U}{\partial a} = \frac{U_2 - U_1}{a_2 - a_1} \qquad (3)$$

where U_2 and U_1 are the strain energies associated with the crack lengths a_2 and a_1. A simple approach to determine G is to perform two finite element analyses, one for the given crack length and the other for an infinitesimally extended crack. The strain energies for the two configurations are used to calculate G. The SIF is then obtained using the relationship between G and K.

G can be also derived using the stiffness derivative approach (Parks, 1974), which eliminates the need for two such finite element runs. One such procedure for both in-plane and out-of-plane crack extensions is discussed by Maiti (1990) (Fig. 2a). G can be expressed as the potential energy release rate

$$G = -\frac{\partial \pi}{\partial l} = \frac{1}{2}\mathbf{u}^\mathrm{T}\frac{\partial \mathbf{K}}{\partial l}\mathbf{u} \qquad (4)$$

where $\mathbf{u}$ and $\mathbf{K}$ are global displacement and stiffness matrices and $\partial\mathbf{K}/\partial l$ is the change in global stiffness matrix per unit crack advance. The crack advance for an in-plane or an out-of-plane extension can be accommodated by rigidly translating the nodes within and on the contour C_0 (Fig. 2a), surrounding the crack tip by an infinitesimal amount ∂l, keeping all other nodal positions unaltered. Hence $\Delta\mathbf{K}$ can be determined by summing up the differences in the stiffness matrices of the elements lying be-

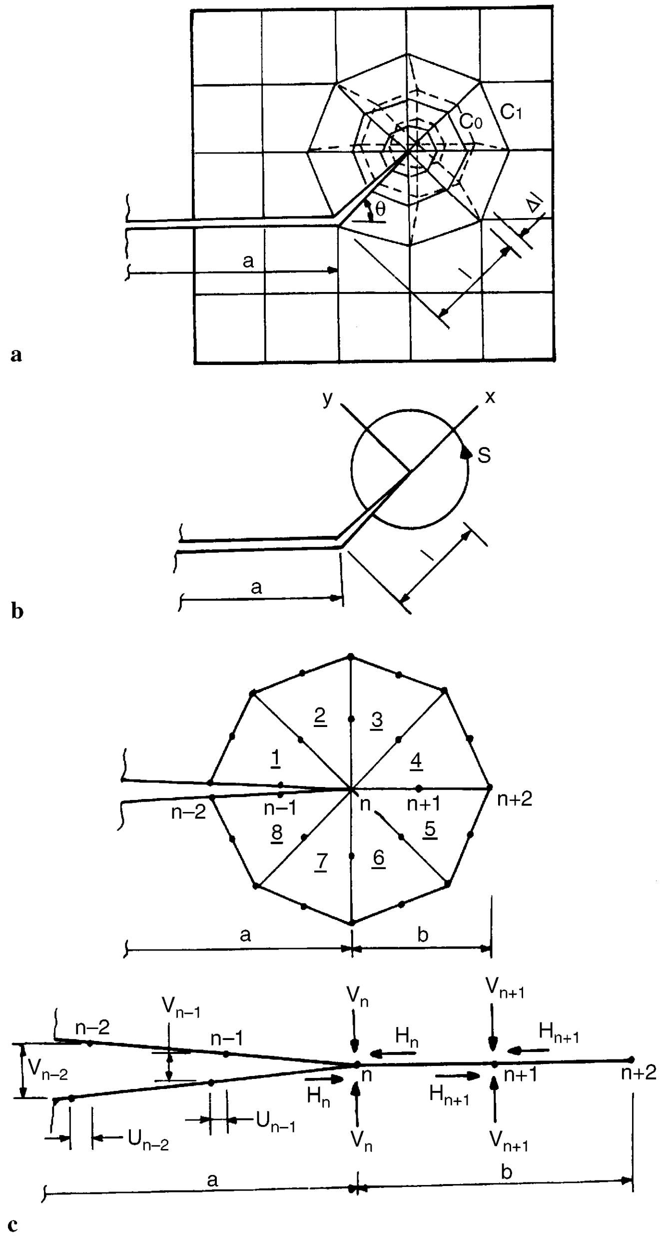


Fig. 2a–c. Arrangements to facilitate calculation of strain energy release rate by **a** stiffness derivative procedure, **b** *J*-integral and **c** modified crack closure integral

tween the contours C_0 and C_1. G is then computed using (4) after solving the set of equations $\mathbf{Ku} = \mathbf{F}$, which corresponds to the initial crack configuration.

In an actual finite element implementation it is easier to calculate $\frac{1}{2}\mathbf{u}^T\Delta\mathbf{K}\mathbf{u}$ as the difference in the total strain energies of the elements, which lie between the contours C_0 and C_1, before and after the shift Δl.

This method has not so far been employed in the BEM.

3.3 *J*-integral

One of the important method which has been used extensively in evaluating the SIFs is based on *J*-integral approach. *J*-integral (Rice, 1968) is defined as follows.

$$J = \int_S (W\,\mathrm{d}x - \mathbf{t}_i \partial \mathbf{u}_i / \partial x)\mathrm{d}S \tag{5}$$

where S is a path (Fig. 2b) surrounding the crack tip, $\mathbf{t}_i$ is the traction vector along the outward normal to the contour, $\mathbf{u}_i$ is the displacement vector, x is the rectangular coordinate aligned with the crack axis and W is the strain energy density. The *J*-integral is equal to the strain energy release rate (SERR) G in the LEFM regime. G can be calculated from J for an in-plane or an out-of-plane crack extension (Fig. 2b). The path-independence of J has been exploited to evaluate the SIFs in both two and three dimensional applications.

The *J*-integral has also been adapted in the BEM for the evaluation of the SIFs. Karami and Fenner (1986) and others have considered the *J*-integral approach to extract the SIFs. In the case of a multidomain analysis the contour for the *J*-integral passes through the two subdomains. The results due to Karami and Fenner (1986) are shown in Fig. 3. Figure 3b shows a typical discretisation and the three circular contours considered for the evaluation of the *J*-integral. The elements are located not only on the crack faces but also on the line ahead of the crack, which is considered to divide the domain into two subregions. The normalised SIF $(K_I/\sigma_0\sqrt{\pi a})$ based on *J*-integral and displacement method, when the quarter point elements are employed around the crack tip, are plotted (Fig. 3c) against the relative crack length a/W. The SIF is also evaluated using the quadratic elements around the crack tip and the *J*-integral (Fig. 3c) to study the effect of shifting of mid side nodes. Karami and Fenner (1986) have indi-

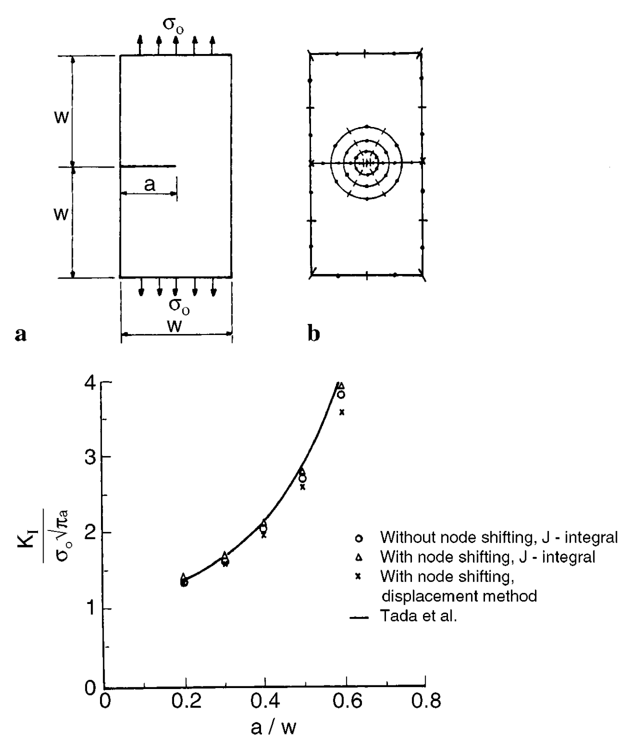


Fig. 3a–c. Edge crack under tensile loading. **a** Plate geometry, **b** mesh and circular contours for evaluation of *J*-integral and **c** variation of SIF with crack length (Karami and Fenner, 1986)

cated that an use of the quarter point elements around the crack tip improves the accuracy of the results and the J-integral approach is more accurate than the displacement method.

For problems with thermoelastic load, the SIFs too can be computed through the J-integrals (Prasad et al., 1994, 1996). Prasad et al. (1994) have used the DBEM and J-integral approach to evaluate the SIFs. The traction and flux equations are applied at nodes on one of the crack edges (Fig. 4a). To satisfy the continuity requirements of the hypersingular integrals in the traction and flux equations, all the elements on the crack edges are modelled as discontinuous elements (Fig. 4a). Calculation of the J-integral is done by taking a circular contour surrounding the crack tip (Fig. 4b). The contour S and domain A are divided into an odd number of linear and triangular segments. The parameters required for the J-integral are calculated at the internal points through the appropriate steps of the BEM. The contour integral is obtained numerically through the trapezoidal rule. The SIF of a rectangular plate with a centre crack (Fig. 4c) under two different boundary conditions is calculated using this

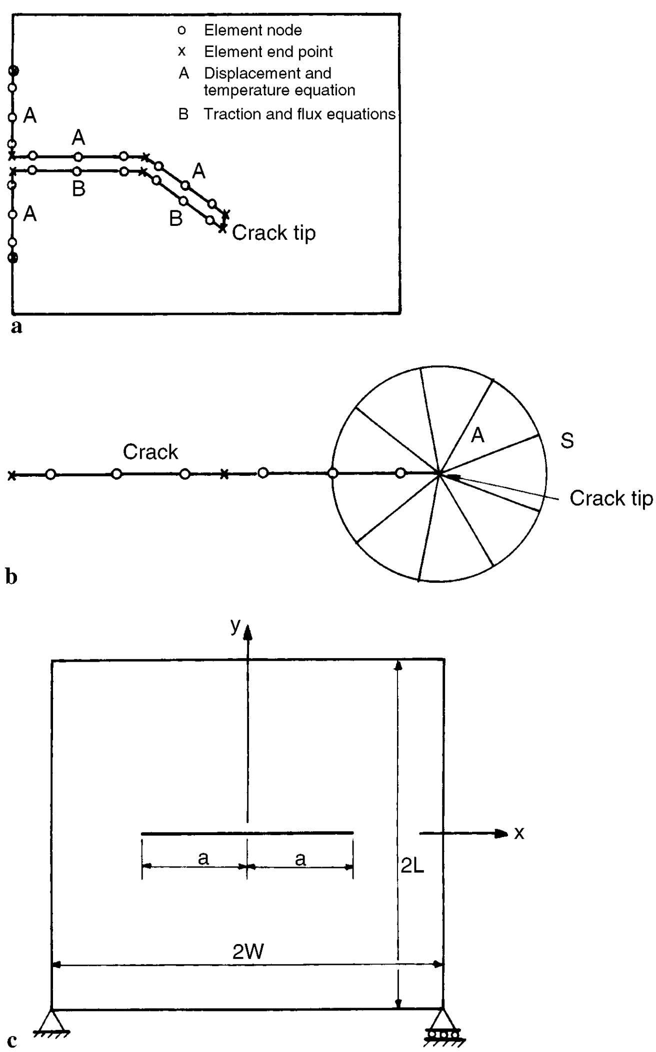


Fig. 4. **a** Modelling of crack for analysis by DBEM. **b** Contour for J-integral. **c** Plate with centre crack (Prasad et al., 1994)

Table 1. Comparison of SIFs for centre crack under mode I and mode II thermal loading

a/W	Stress intensity factor					
	Sumi et al. (1980)		Prasad et al. (1994)		Mukhopadhyay et al. (1999b)	
	Mode I	Mode II	Mode I	Mode II	Mode I	Mode II
0.1	0.2750	0.021	0.268	0.018	0.2697	0.0197
0.2	0.3499	0.053	0.347	0.054	0.3461	0.0527
0.3	0.4060	0.094	0.401	0.095		0.0933
0.4	0.4599	0.141	0.448	0.141	0.4489	0.1378
0.5	0.4900	0.188	0.491	0.190	0.4903	0.1855
0.6	0.5249	0.247	0.525	0.243	0.5241	0.2380

approach. The boundary conditions for a pure mode I problem are $\theta = \theta_1$ on the crack and $\theta = \theta_2$ around the boundary. Second boundary conditions are pure mode II: $q = 0$ on the crack; $q = 0$ at $x = \pm W$, $|y| < L$; and $\theta = \pm\theta_2$, $|x| < W$, $y = \pm L$. Here $\theta_1 = 0°C$ and $\theta_2 = 10°C$. The calculated SIFs (Prasad et al., 1994) are shown in Table 1. The results are path independent and compare well with other reported results.

In the J-integral approach additional computations of displacement and traction at internal points is needed. This stands a disadvantage of this method.

3.4 Modified crack closure integral

Another important methods of determining the SIF is based on the crack closure integral (CCI) technique. The CCI is first introduced by Irwin (1958). In the FEM, the adoption of the concept of CCI have contributed to a significant improvement of accuracy of the SIFs over the computation based on the displacement method. The method can be applied to problems of mixed mode loading as well. This technique is first adopted in the FEM by Rybicki and Kanninen (1977). They described the method considering a linear variation of the displacement field around the crack tip. Consequently, the element ensures a constant strain field. They termed the method as modified crack closure integral (MCCI) technique. Later it has been shown by Krishnamurthy et al. (1985), Ramamurthy et al. (1986), Sethuraman and Maiti (1988), Maiti (1990, 1992b) and Badri Narayan (1994) that crack line displacement and stresses can be locally smoothed using the computed nodal data. These smoothed field can then be employed to compute the crack closure work. This procedure gives the energy release rate, and hence the SIFs, which have very high accuracy than the results based on the displacement method. Chao and Atluri (1995) and Chao et al. (1995) have presented the calculation of SIFs for interfacial crack using virtual crack closure integral (VCCI) and interfacial crack in dissimilar anisotropic media using hybrid element and mutual integral. Recently, Singh et al. (1998) have proposed an universal crack closure integral (UCCI) which is independent of the basic stress analysis procedure.

For an in-plane extension (Fig. 2c) the finite element equivalent of the MCCI (Maiti, 1990) is

$$
\begin{aligned}
G &= \operatorname*{Lt}_{b\to 0} \frac{1}{2b}\left[H_n u_{n-2} + H_{n+1} u_{n+1}\right] \\
&\quad + \operatorname*{Lt}_{b\to 0} \frac{1}{2b}\left[V_n v_{n-2} + V_{n+1} v_{n+1}\right] \\
&= G_{\mathrm{I}} + G_{\mathrm{II}} \qquad (6)
\end{aligned}
$$

where b is the virtual crack extension, H_n, V_n, H_{n+1}, and V_{n+1} are the crack closure forces. u_{n-2}, u_{n-1}, v_{n-2} and v_{n-1} are the opening displacements. G_{I} and G_{II} are the components of mode I and mode II energy release rates. The nodal forces H_n, V_n, etc., are obtained from the element nodal forces of element 1, 2, 3 and 4 (or 5, 6, 7 and 8) in Fig. 2c. The opening displacements are taken approximately equal to those of nodes $n-1$ and $n-2$ corresponding to the original crack. Thereby it is possible to avoid a computation of the opening displacements for the extended crack. For an out-of-plane extension of crack also G can be evaluated in a similar manner (Maiti, 1990).

Recently, few investigators, e.g. Hucker and Farris (1993), Farris and Liu (1993), Mukhopadhyay et al. (1998a, b, 1999a), etc., have adapted the MCCI based procedure in the BEM for the evaluation of the SIFs. When a crack is under remote mechanical loading, the unknown tractions over the ligament gives rise to a nodal traction at the crack tip node (Fig. 5). However, these crack tip tractions do not contribute to any loading over the span OA (Fig. 5). For a traction free crack edge, while evaluating $\int Ut\, d\Gamma$, where U is the fundamental solution for displacement and t is the specified traction, over the portion OA, it can be taken as zero irrespective of the presence of the crack tip traction at the crack tip. The crack closure work can be calculated with a similar presumption; thereby the contributions come only from the element OB on the ligament side. The crack closure work is accordingly given by,

$$
W_{\mathrm{I}} = \frac{1}{2}\int_0^l vt\, dx \ , \qquad (7)
$$

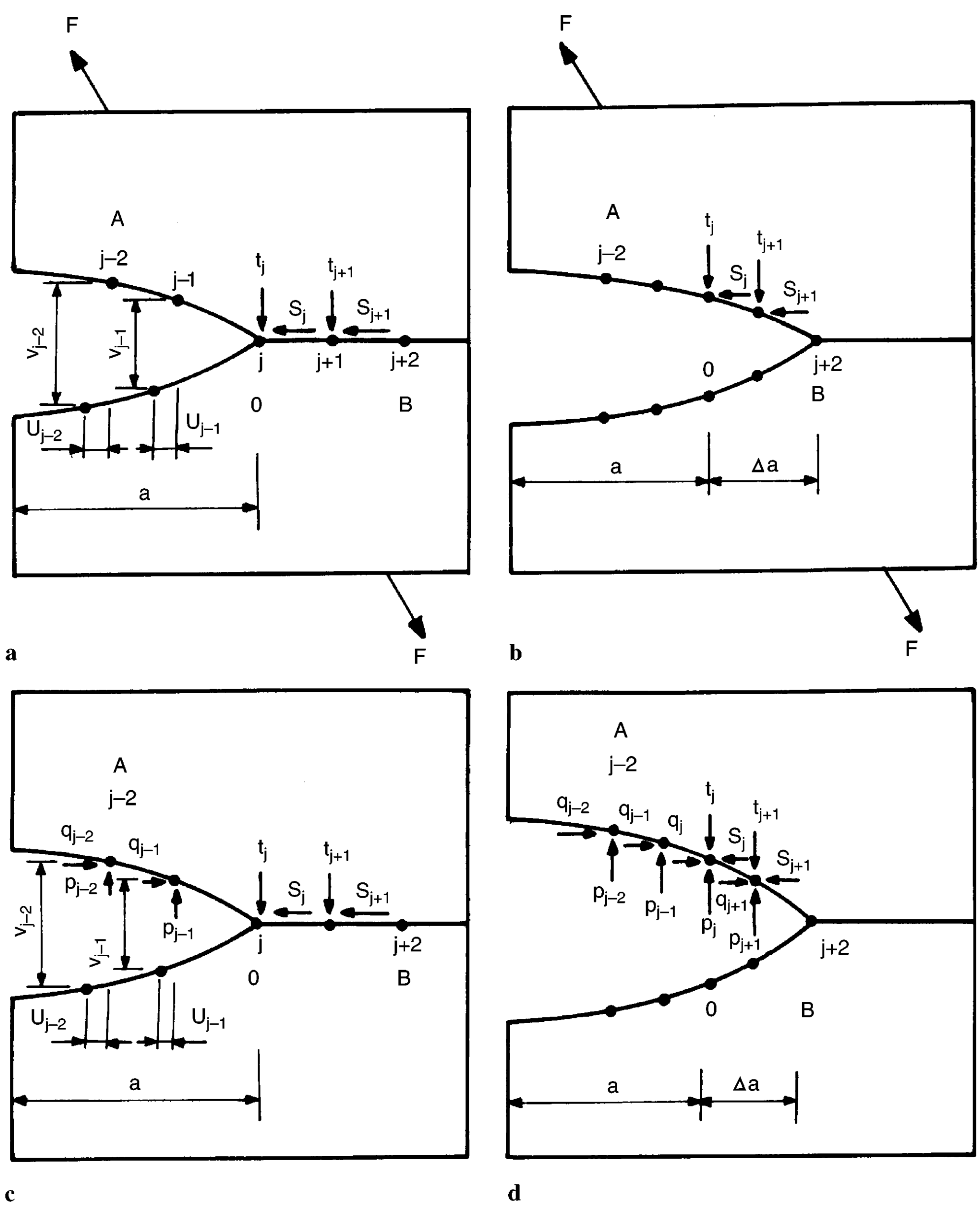


Fig. 5a–d. Illustration of crack closure forces. **a** Remote loading and **b** closure forces, **c** crack edge loaded externally and **d** closure forces and external loading

In the case of quadratic elements around the crack tip (Fig. 5) the displacement variation over OA is given by

$$v = v_{j-1} - 0.5\, v_{j-2}\, \xi + (0.5\, v_{j-2} - v_{j-1})\xi^2 \quad (8)$$

where ξ is the natural coordinate. Similarly the traction variation, which is also quadratic, has the form

$$t = t_{j+1} + 0.5(t_{j+2} - t_j)\xi + [0.5(t_{j+2} + t_j) - t_{j+1}]\xi^2 \quad (9)$$

The mode I strain energy release rate G_{I}

$$G_{\mathrm{I}} = [v_{j-1}(t_j c_1 + t_{j+1} c_2 + t_{j+2} c_3) + v_{j-2}(t_j c_4 + t_{j+1} c_5 + t_{j+2} c_6)]/60 \quad (10)$$

where the coefficients are defined below.

$c_1 = 2;\ \ c_2 = 16;\ \ c_3 = 2;\ \ c_4 = 4;\ \ c_5 = 2;\ \ c_6 = -1$

A similar expression for G_{II} can be obtained involving x-component of tractions and displacements. Similar correlations for constant element are available in Hucker and Farris (1993) and for linear and quarter point elements in Mukhopadhyay et al. (1998a). In Farris and Liu (1993) expressions for SERRs for three modes G_{I}, G_{II} and G_{III} are available for 8-node quadrilateral BEs for application in 3-D.

Loading on the crack edges come up due to explicit mechanical loading, e.g., crack subjected to fluid pressure, etc. In such a case, when a crack extends or an extended crack is closed, it must be noted that there is an extra loading on the newly formed crack edges on top of the usual crack closure forces. This loading contributes to an additional amount of work. A crack closure integral calculation must therefore cognise this fact.

In the case of mechanical loading applied away from the crack edges (Fig. 5a), the nodes $j-2$ and $j-1$ on the crack edge AO are free of any load. As the crack extends up to B, the newly formed crack edges are also load-free (Fig. 5b). If the crack edges are subjected to, say, fluid pressure (Fig. 5c), as the crack extends, the newly formed crack edges are also subjected to the same fluid pressure (Fig. 5d). The crack edges will therefore undergo an extra opening. The crack closure work have two parts. One part is due to the usual tractions t_j, t_{j+1} and t_{j+2} and the other part is due to fluid pressure p. For a mode I problem the crack closure work is given by

$$W_{\mathrm{I}} = \frac{1}{2}\int_0^l vt\,\mathrm{d}x + \frac{1}{2}\int_0^l vp\,\mathrm{d}x \quad (11)$$

where p is intensity of distributed crack edge normal load and v is full opening displacement. The direction of this load is positive, when it produces effects additive to that due to the external load. The pressure loading can be of uniform intensity or with a linear or quadratic variation. Correlation of SIFs for linear, quadratic and quarter point elements for crack edges subjected to constant 'fluid' pressure are due to Mukhopadhyay et al. (1998b).

The treatment of thermal load in the MCCI based calculations are given by Mukhopadhyay et al. (1999a). For a mixed mode thermal stress problem the domain is to be broken into a convenient number of subregions. At each node of the common interface of the two adjacent regions temperature and displacement are the same; temperature gradients and tractions are equal and opposite. The crack edges are loaded by thermal loads. As the crack tip advances, the newly formed crack edges give way to crack opening but the traction conditions are not changed. Before and after the crack extension the temperatures and gradients are the same. Hence the mode I crack closure work for a crack subjected to a remote mechanical plus thermal loading can be calculated as per Eq. (7). When a problem involves remote as well as fluid pressure on the crack edges/faces, on top of any thermal loading, as the crack extends, the newly formed crack edges become subjected to the same fluid pressure. The newly formed crack edge will also be subjected to traction due to the thermal loading. This traction remains unchanged throughout the crack opening. So it will not contribute to the crack closure work. The fluid pressure on the extended crack edge only contributes an extra work to the closure work. Thus, in the case of thermal loading over the whole domain and fluid pressure on the crack edges the crack closure work has two parts. One part is due to the closure forces arising out of thermal loading and fluid pressure and the other is due to crack edge loading p on the extended crack. The crack closure work can be calculated as per Eq. (11). It must be noted here that, in this context, there is a marked difference between the BEM and FEM. In the FEM, as has been shown by Maiti (1992b), the work calculation involves extra contribution from the thermal loading too.

The accuracy of the MCCI based computation of SIFs have been demonstrated by several examples by Hucker and Farris (1993), Farris and Liu (1993) and Mukhopadhyay et al. (1998a, b, 1999a). One of the cases studied by Hucker and Farris (1993) is presented here. They have analysed a centre cracked plate under uniformly distributed load employing constant elements (Fig. 6a). The mode I SERR G_{I} is determined using the near-tip crack opening and tractions. They have compared the SIFs based on displacement, stress and MCCI methods (Fig. 6b). In the case of the displacement and stress methods, the results are based on data obtained from two elements off the crack tip. A considerable scatter is observed in the case of displacement and stress methods when near-tip displacement and stresses are used (Fig. 6c). The MCCI based results are in excellent agreement with the reference solution and the scatter is negligible even though crack tip element size Δa is 20–25% a. Farris and Liu (1993) have demonstrated that for 3-D crack problems, in general, adapting a ratio of the width of crack front elements to the crack depth 1/10, the error in the SIF evaluation is limited to ±1.5%.

4
Singularity elements

The main difficulty in modelling fracture problems arises from the presence of a stress singularity at the crack tip. Most of the singularity elements available in the general purpose packages are based on the displacement formulation. Some use the displacement variation in keeping with the singularity, others get it through an appropriate mapping or transformation.

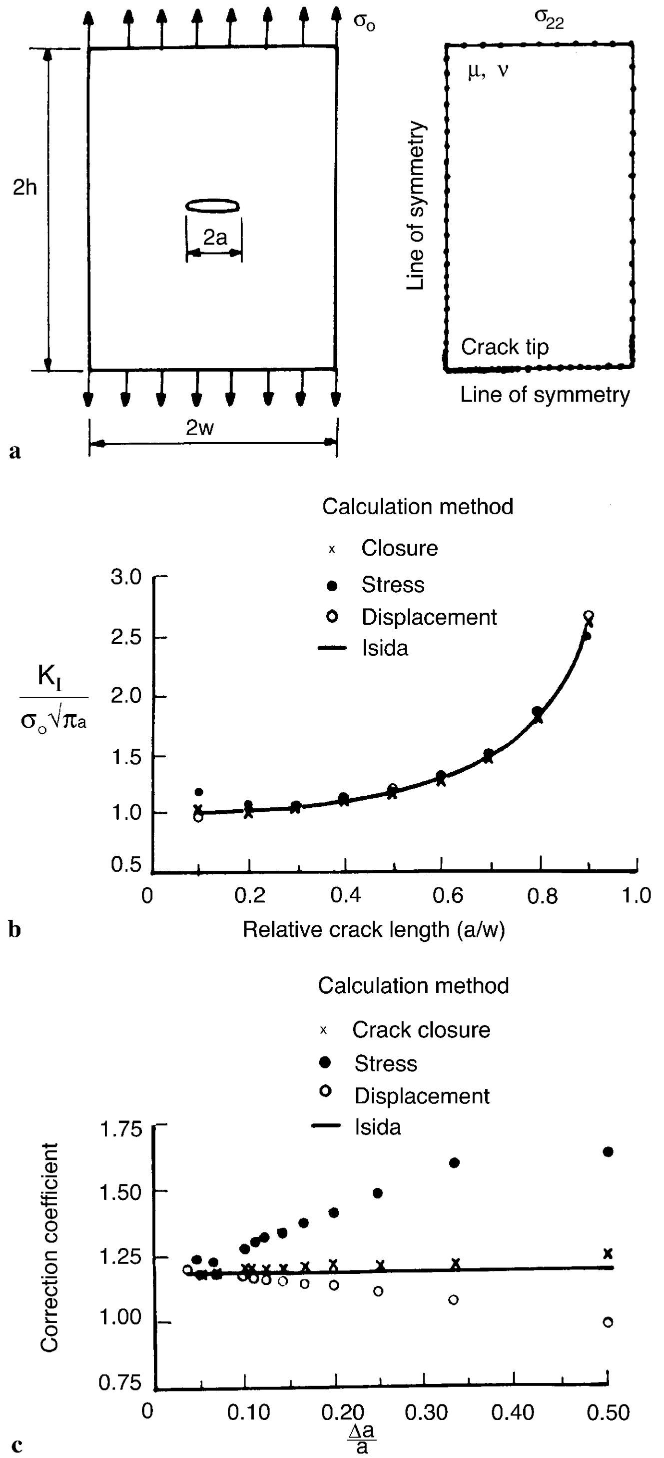


Fig. 6a–c. Centre crack under tensile load. **a** Geometry and mesh. **b** Variation of SIF correction factor with crack length. **c** Effect of crack tip element size (Δa) on accuracy of SIFs for $a/W = 0.5$ (Hucker and Farris, 1993)

4.1 Square root singularity

Around an elastic crack tip there are both strain and stress singularities. When the FEM is used to analyse a crack problem, the satisfaction of strain singularity automatically ensures the stress singularity. In FEM, several singularity elements, e.g. Hensell and Shaw (1975), Tracey and Cook (1977), Barsoum (1977), etc., exist which can incorporate the square root strain singularity. The element due to Barsoum (1977), also known as quarter point element, is based on mapping technique. He has shown that a simple shift of the mid node to the quarter position gives rise to a square root strain singularity at the neighbouring corner node. This element has been extensively used and its performance has been widely reported.

While using the analogue of the wellknown quarter point finite element (Barsoum, 1977) in the BEM, the assumption of a variation of displacement in proportion to square root of distance from the crack tip does guarantee a square root strain singularity. This cannot however automatically ensure a square root traction singularity. Because of this the quarter point element is sometimes termed as 'strain singularity element' in the BEM. This element therefore enables a partial modelling of the singularities.

The issue of partial and total modelling of singularities have received a considerable attention in the BEM. A large number of investigators, e.g. Cruse and Wilson (1978), Tan and Fenner (1979), Xanthis et al. (1981), Blandford et al. (1981), Nadiri et al. (1982), Weeen (1983), Martinez and Dominguez (1984), Aliabadi et al. (1987, 1989), Gangming and Yongyuan (1988a, b), Zang and Gudmundson (1990), Chandra et al. (1995), Saez et al. (1995), Chen and Chen (1995), Watson (1995), etc., have contributed in this direction. Cruse and Wilson (1978), Tan and Fenner (1979), Nadiri et al. (1981), etc., have used only the strain singularity element and showed the improvement in the accuracy of the SIFs over the results through the nonsingular elements. Blandford et al. (1981) introduced a special crack tip element which ensured both the strain and traction singularities. In this element, the mid-node is shifted to the quarter point to ensure the strain singularity. The displacements are expressed as the following (Fig. 7a).

$$u_i = A_i^1 + A_i^2\sqrt{r} + A_i^3 r \tag{12}$$

The displacement variation is analytically correct for the first terms of the infinite series expansion. However if the

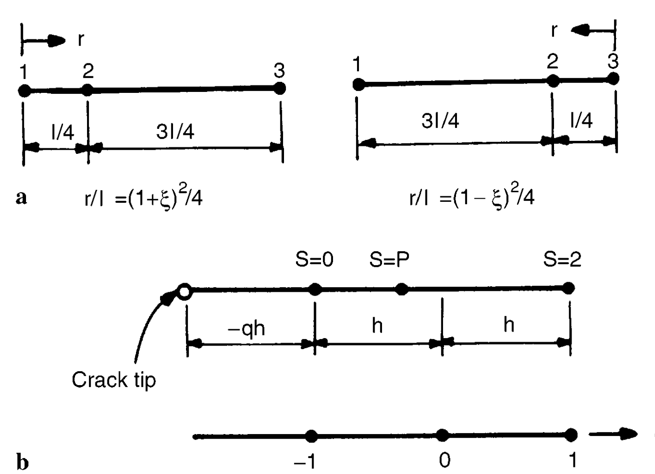


Fig. 7a, b. Singularity element due to Blandford et al. (1981). **a** Traction singular quarter-point element. **b** Transition element coordinate mapping same variation is assumed for the traction, it does not demonstrate any singularity. This is because the function consists of only $\sqrt{r}$ and r terms. They introduced a new function to define traction variation.

$$\begin{aligned} t_i &= \left(A_i^1 + A_i^2\sqrt{r} + A_i^3 r\right)\sqrt{(l/r)} \\ &= B_i^1/\sqrt{r} + B_i^2 + B_i^3\sqrt{r} \end{aligned} \tag{13}$$

This type of a function is capable of giving the correct traction variations in the vicinity of the crack tip ($r \to 0$). Thus the displacements and traction variations for the elements around the crack tip are given by Eqs. (12) and (13). The resulting boundary element is a 'traction singular quarter point boundary element' (Blandford et al., 1981). They have also demonstrated the usefulness of transition elements. These elements (Fig. 7b) help to extend the distance over which the presence of the crack is felt. The three nodes of the transition element, designated by nondimensional variable $s = 0$, p, 2 have been mapped on the ξ-axis. Lynn and Ingraffea (1978) have shown that by placing the midpoint node at $p = [(1-q) + \sqrt{(q^2+2q)}]/2$, the correct stress singularity is obtained after differentiation of the element shape functions. The technique, i.e. multiplication by $\sqrt{(l/r)}$, used to transform the quarter point boundary element into a traction singular quarter point boundary elements, is not applicable to the transition element because this will affect the continuity of the traction modelling. Blandford et al. (1981) have commented that the transition element must be tested completely to ensure their effectiveness.

Saez et al. (1995) have presented effective implementation for SIF computation of a mixed BE approach based on the standard displacement integral equation and the hypersingular traction integral equation. Expressions for the evaluation of the hypersingular integrals along general curved cracks quadratic line elements are presented. Discontinuous elements are used to satisfy the C^1 continuity requirement of the hypersingular integral equations. The generality of the method allows for the modelling of curved cracks and the use of straight line quarter point elements. The elements may have any general quadratic geometry defined by three nodes and the collocation points can be set at any position within the element. It may be noted that in the context of hypersingular formulation, Saez et al. (1995) demonstrated that straight line quarter point element can be used at the vicinity of the crack tip to produce the $\sqrt{r}$ displacement variation along the crack

Watson (1995) employed the Hermitian cubic shape functions to characterise the singularity for straight and curved cracks under a plane strain condition. The cartesian coordinates of an arbitrary point of the element S_b (Fig. 8a) are the following.

$$x_i(\xi) = \sum_{c=1}^{2}[M_c(\xi)x_i(b,c) + N_c(\xi)s_i(b,c)] \tag{14}$$

where $M_c(\xi)$ and $N_c(\xi)$ are Hermitian cubic shape functions, $x_i(b,c)$ are the cartesian co-ordinates of node c of element S_b and $s_i(b,c)$ is a vector tangent to S_b at node c. Displacement and traction are given by

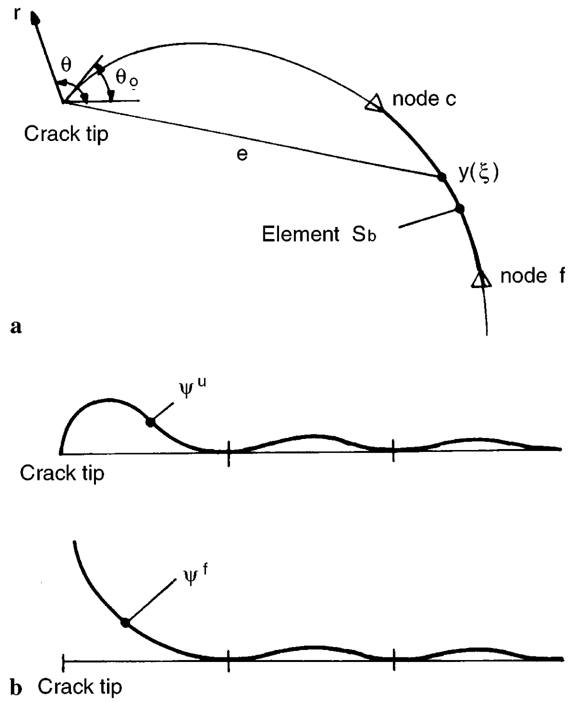


Fig. 8a, b. Singularity element due to Watson (1995). **a** Element S_b on crack. **b** Extended singular shape functions

$$\begin{aligned} u_i(\xi) &= \sum_{c=1}^{2}\left[M_c(\xi)u_i(b,c) + N_c w_i(b,c) \right. \\ &\qquad \left. + \sum_{k=1}^{4}\Psi^u_{bcik}(\xi)\phi_k(b,c)\right] \\ t_i(\xi) &= \sum_{c=1}^{2}\left[M_c(\xi)t_i(b,c) + N_c v_i(b,c) \right. \\ &\qquad \left. + \sum_{k=1}^{4}\Psi^t_{bcik}(\xi)\phi_k(b,c)\right] \end{aligned} \tag{15}$$

where $u_i(b,c)$ and $t_i(b,c)$ are displacement and traction at node c of element S_b, $w_i(b,c)$ and $v_i(b,c)$ are derivatives with respect to ξ of displacement and traction at that node, $\Psi^u_{bcik}(\xi)$ and $\Psi^t_{bcik}(\xi)$ are singular shape functions. $\phi_1(b,c)$ and $\phi_2(b,c)$ correspond to the dominant symmetric and antisymmetric crack opening modes, whereas $\phi_3(b,c)$ and $\phi_4(b,c)$ correspond to the subdominant symmetric and antisymmetric crack opening modes. The singular shape functions constructed from the terms of the eigen function expansion are available in Watson (1995). They extend over many elements to either side of the crack tip. They are terminated at a corner of the boundary, the other crack tip for a buried crack, or at a node for which the angle between the tangent to the boundary and the limiting tangent at the crack tip exceeds a threshold value. The singular shape functions over three elements are shown in Fig. 8b. The functions have zero value and zero derivative at all the nodes except for the node at the crack tip.

Recently, Kebir et al. (1999) have analysed mixed-mode crack growth in bolted joints using the DBEM. All the boundaries are discretised with discontinuous quadratic

BEs and the crack tip is modelled by singular elements that exactly represent the strain field singularity $1/\sqrt{r}$. The nodes are positioned at $\xi = -2/3$, $\xi = 0$ and $\xi = 2/3$. The shape functions of this element are (Kebir et al., 1999)

$$N_1(\xi) = \frac{3}{2}\frac{(3-\sqrt{15})\xi + 2\sqrt{1+\xi} - 2}{\sqrt{15}+\sqrt{3}-6}$$
$$N_2(\xi) = \frac{3(\sqrt{15}-\sqrt{3})\xi - 12\sqrt{1+\xi} + 2(\sqrt{15}+\sqrt{3})}{2(\sqrt{15}+\sqrt{3}-6)} \quad (17)$$
$$N_3(\xi) = \frac{3}{2}\frac{(\sqrt{3}-3)\xi + 2\sqrt{1+\xi} - 2}{\sqrt{15}+\sqrt{3}-6}$$

It is evident that the above formulation represents exactly the strain singularity because $\partial N_l/\mathrm{d}\xi = \infty$ at $\xi = -1$. The improper integrals that arise in the dual integral equations are handled analytically. Aliabadi et al. (1987, 1989) proposed a strategy, whereby an analysis is possible by removing the stress singularity at the crack tip.

In a heat conduction problem too there are singularities at the crack tip as has been discussed by Emery et al. (1977) and Chao and Chang (1992). Emery et al. (1977) has pointed out that when the flow of heat is interrupted by a free boundary, e.g. crack edge, the field shows a singularity near the crack tip. The near field solution of temperature around the crack tip is given by

$$\phi(x,y) = K_T r^n \sin n\theta \quad (18)$$

where $n = 1/2$, 1, 3/2 etc. Here $\phi(x,y)$ denotes the temperature and K_T is the coefficient of thermal singularity. Obviously $n = 1/2$ indicates a singularity in $\partial\phi/\partial r$ and heat flux at the crack tip. When the analysis is performed using the FEM, the temperature derivative $(\partial\phi/\partial r)$ singularity is only to be ensured. This may be achieved by employing the quarter point elements around the crack tip and one need not really bother separately about the singularity in the normal derivative/heat flux $(\partial\phi/\partial n)$. However when the BEM is employed, singularities in both the radial derivative $(\partial\phi/\partial r)$ and normal derivative $(\partial\phi/\partial n)$ are to be handled for a total modelling of the singularity. In the BEM, an use of the quarter point element can ensure a partial modelling, i.e. the singularity in the temperature derivative is only taken care of. The effect of simulating both the temperature derivative and heat flux singularities on the computation of SIFs using the BEM has been discussed by Katsareas and Anifantis (1995), Prasad et al. (1996), Katsareas et al. (1998) and Mukhopadhyay et al. (1999b). Katsareas and Anifantis (1995) and Katsareas et al. (1998) have employed the traction singular quarter point element along with multiregion technique; Prasad et al. (1996) have adapted the discontinuous quarter point element and the DBEM.

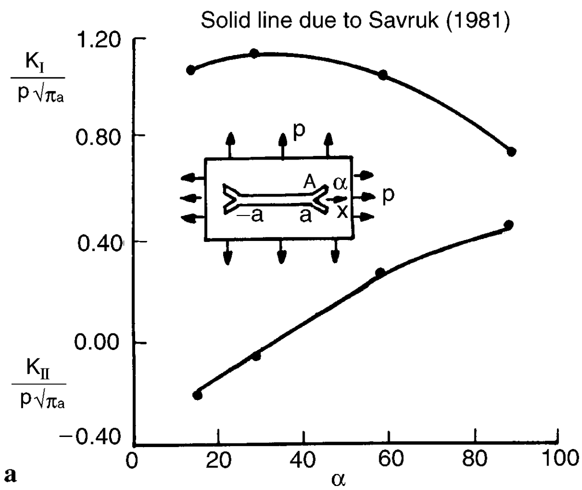
While evaluating the SIFs employing the special crack tip elements, the displacement method is again the most widely employed technique. As an example, Kebir et al. (1999) have evaluated the SIFs from crack opening displacements at collocation points extremely close to the crack tip. Prasad et al. (1994) have used the J-integral. The effect of partial and total modelling of singularities on computation of SIFs through the MCCI method is presented in the case of mechanical and/or thermal loading by Mukhopadhyay et al. (1999a). The computed SIF correction factors based on the MCCI method for a centre crack under mode I and mode II thermal loads are compared (Table 1) with the analytical solutions of Sumi and Katayama (1980). It is also shown that the MCCI based computation of SIFs offers flexibility in the selection of size of the crack tip element. When the crack tip is surrounded by traction/heat flux singularity elements, there is a product of two singularity terms. In such situations a higher order of Gaussian quadrature (8 to 10) is recommended. In the case of thermal stress problems there is a possibility of cumulative effect of the partial or total modelling, since the heat conduction analysis is followed by the stress analysis.

4.2
Variable order singularity

There are situations where the order of singularity at a point in a given domain is variable. In the case of stress analysis, a kinked crack is a typical example. As has been shown by Williams (1952) the order of singularity at the knee varies with the knee angle. Cook and Erdogan (1972) have indicated that for a crack terminated at a bimaterial interface, the order of singularity at the crack tip lying on the interface varies with the material combinations and the state of stress, i.e. plane stress or plane strain. Lo (1978) and Cotterell and Rice (1980) have presented analytical solutions for kinked cracks. There are a host of finite elements, e.g. Tracey and Cook (1977), Stern (1979), Maiti (1992c), etc., just to mention a few, which can help to model such problems. These elements are based on either the displacement or hybrid formulation (Pian et al., 1971, Atluri et al., 1975). The hybrid formulation is mathematically elegant, permits direct computation of SIFs and provides high accuracy even when a small number of such elements is used around the crack tip. The displacement formulation is more simple and widely used.

Wang and Chau (1997) have proposed a procedure for calculating the interaction between cracks and holes. Singular interpolation function of order $1/\sqrt{r}$ are introduced for the discretisation of the crack near the crack tip, such that stress singularity can be modelled appropriately. The singular integrands involved at the element level are integrated analytically. For each of the hole boundaries, an additional unknown constant is introduced such that the displacement compatibility condition can be satisfied exactly by the complex boundary function $H(t)$, which is a combination of the traction and displacement density (Wang and Chau, 1997). An attractive feature of the method is that the SIFs can be derived analytically in terms of the crack unknown $H(t)$. The interaction between a straight crack and a circular hole and a kinked crack and a circular hole is demonstrated. However while modelling the kinked crack, a large number of elements (e.g. 130 linear elements) is employed. Wang and Chau (1997) opine that this procedure can be adapted for solving elastic bodies containing different number, distribution, orientations and shapes of holes and cracks.

Mogilevskaya (1997) has reported numerical modelling of kinked cracks and 2-D smooth crack growth. He has



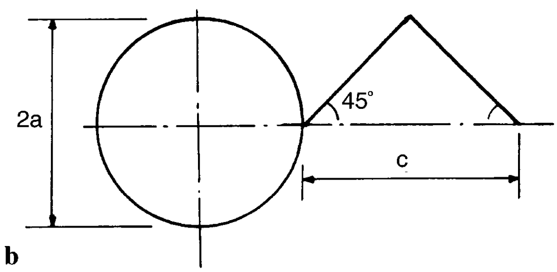


Fig. 9. **a** Variation of SIF with angle α for crack with four kinks. **b** Kinked crack emanating from circular hole (Mogilevskaya, 1997)

adapted analytical formulae for calculation of the SIFs. The displacement discontinuities (DD) involved in these formulae are found from the numerical solution of a complex hypersingular integral equation (CHSIE) for a piecewise homogeneous plane with curvilinear cracks. One of the examples solved by him deals with four kinks under biaxial stress (Fig. 9a). The main crack is represented by four boundary elements and each kink is represented by one element. The computed SIF correction factors are observed by him to be in good agreement with the reference solutions. Another example considered by Mogilevskaya deals with a tension of a plane with a kinked crack emanating from a circular hole (Fig. 9b). Two cases of loading uniaxial and biaxial are considered. The normalised SIFs are compared with the reference solutions from Murakami (1987) in Table 2. He has indicated that the CHSIE method gives reliable SIFs.

Recently Mukhopadhyay et al. (1999c) have proposed two special BEs which can model variable order singularities. The element, which models only the strain (radial temperature derivative, $\partial\phi/\partial r$) singularity is termed as the Variable Strain Singularity (VSS) element. Similarly, the element which incorporates both the strain and traction (normal heat flux, $\partial\phi/\partial n$) singularities is termed as the Variable Strain and Traction Singularity (VSTS) element. These elements have been developed by a simple manipulation of the element shape functions. The set of shape functions for the VSS element are

$$N_1 = 2^c[-(r/l)^c + (r/l)^{c+1}] + (r/l)$$
$$N_2 = 2^{c+1}[(r/l)^c - (r/l)^{c+1}] \quad (18)$$
$$N_3 = 2^c[-(r/l)^c + (r/l)^{c+1}] - (r/l) + 1$$

These shape functions ensure a slope of infinity at one end node of interest. The shape functions also fulfil the rigid body and constant strain criteria. This element partially models the variable order singularity; it ensures a strain singularity as $r \to 0$, but not the traction singularity. To model the singularity behaviour totally, separate shape functions are needed to incorporate the variable traction singularity. A set of such shape functions are given below.

$$M_1 = 2^c[(l/r)^{1-c} - (r/l)^c] + 2(r/l) - 2$$
$$M_2 = 2^c[(l/r)^{1-c} - (r/l)^c] \quad (19)$$
$$M_3 = 2^c[-(l/r)^{1-c} + (r/l)^c] + 1$$

These shape functions are obtained with the help of the derivatives of the set of shape functions (18). A simultaneous representation of displacement and traction fields by Eq. (18) and Eq. (19) respectively gives a VSTS element. This can perform total modelling of variable order singularities.

4.3
Neighbouring singularities

The presence of microcracks can severely affect the stress concentration around a macrocrack. Studying the effect

Table 2. SIF correction factor for a kinked crack emanating from a circular hole in a plane

a/R	SIF correction factor Y							
	Isida et al. (1984)		Mogilevskaya (1997)		Mukhopadhyay et al. (1999d)			
					TESS		TESTS	
	Y_I	Y_{II}	Y_I	Y_{II}	Y_I	Y_{II}	Y_I	Y_{II}
Uniaxial tension								
0.1	2.0920	−1.0340	2.0940	−1.0465	2.1173	−1.0532	2.1341	−1.0545
0.2	1.8030	−0.8660	1.8065	−0.8757	1.8277	−0.8744	1.8389	−0.8733
0.5	1.3140	−0.6240	1.3175	−0.6319	1.3085	−0.6192	1.3218	−0.6179
1.0	0.9700	−0.5060	0.9736	−0.5084	0.9502	−0.4964	0.9551	−0.4975
Biaxial tension								
0.1	1.4930	−0.7540	1.4937	−0.7628	1.5136	−0.7674	1.5210	−0.7691
0.2	1.3620	−0.6750	1.3647	−0.6818	1.3767	−0.6827	1.3898	−0.6819
0.5	1.1270	−0.5310	1.1321	−0.5402	1.1102	−0.5417	1.1252	−0.5404
1.0	0.9450	−0.4190	0.9636	−0.4184	0.9259	−0.4271	0.9361	−0.4256

of neighbouring singularities is crucial as in many practical applications a macrocrack can be surrounded by microcracks. Chudnovsky and Kachanov (1983), Kachanov (1986), Raju (1987), Hori and Nemat-Nasser (1987), Rubeinstein and Choi (1988), Dutta et al. (1990), Lam and Phua (1991), Maiti (1992a), Chen and Hasebe (1995), Mogilevskaya (1997), etc., have analysed the problem of interaction of neighbouring singularities. Depending on the location and size of the microcracks this may lead to a 'shielding' or 'enhancement' effect on the SIF at the main crack tip. The studies by Lam and Phua (1991) is based on a singular integral equation method which uses distributions of edge dislocations to represent a crack in a mathematical model. Chudnovsky and Kachanov (1983), Kachanov (1986), Hori and Nemat-Nasser (1987), Rubeinstein and Choi (1988), etc., have adapted analytical approaches to study crack-crack interactions.

The interaction of neighbouring singularities can be analysed routinely by employing a large number of elements in the FEM or BEM. Zang and Gudmundson (1990) have solved contact problems of kinked cracks by the BIEM. Based on the integral equation for the resultant forces along a crack, a numerical method is developed for the solution of two dimensional kinked crack problems taking crack contact into account. One of the case studies deals with a surface crack with two kinks (Fig. 10a). The half-plane is subjected to a moving uniform pressure P_0 over an interval $2c$ ($c = 1$) on its edge. They have discretised the crack line using 70 linear elements and all together there are 150 degrees of freedom. They have also studied the same problem using the FEM which requires 800 8-noded isoparametric elements. The computed displacement jumps for $P_0/E = 1.2/205$, $\nu = 0.3$ and $d = 0$ by the proposed BIEM and FEM are in good agreement (Fig. 10b). The computed SIFs (normalised with respect to $P_0\sqrt{\pi c}$) as a function of the loading position d are presented in Fig. 10c. The CPU time involved in the BIEM is about 5 h for all 73 loading steps and about 182.5 h when the FEM is employed.

Chen and Chen (1995) while studying multiple cracks problems have commented that the multi-domain approach, as introduced by Blandford et al. (1981), may lead to the formation of many artificial boundaries. As the nodes on the ligament of the cracks are common to the adjacent domains, the number of nodes will also increase. They have opined that, in the DBEM, as proposed by Portela and Aliabadi (1992a) and Portela et al. (1992b), since the displacement integral equation also needs to be established to model the outer boundary and the finite-part integrals are evaluated directly, the difficult and time-consuming task of computation increases, especially for the problem with multiple cracks.

In the BIEM, Weaver (1977), Takakuda et al. (1985a), Takakuda (1985b), Polch et al. (1987), Gray et al. (1990), etc., derived a traction boundary integral equation to analyse different type of cracks existing in an infinite body. The relative displacement derivative (Weaver, 1977; Polch et al., 1987) and relative displacement (Takakuda, 1985a; Takakuda et al., 1985b; Gray et al., 1990) are taken as independent variables in the formulation.

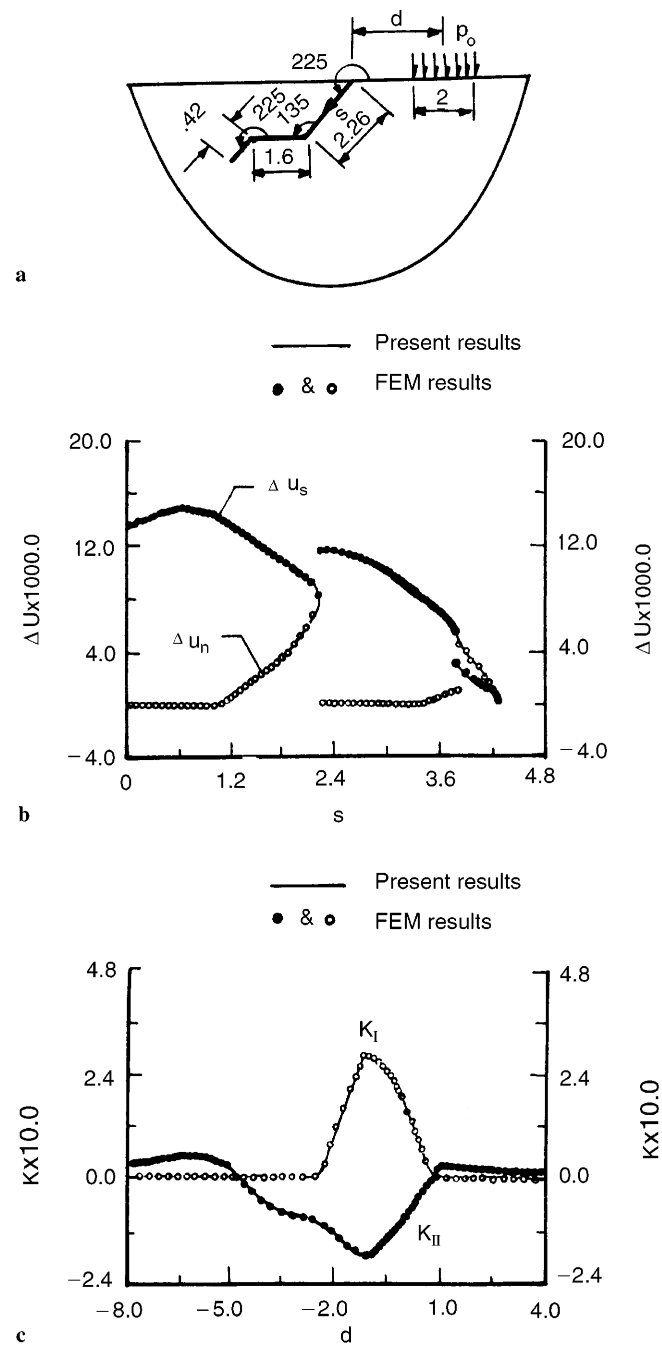


Fig. 10. Edge crack with two kinks. **a** Geometry and loading. **b** Crack displacement jumps. **c** SIFs as a function of load positions (Zang and Gudmundson, 1990)

However the analytical evaluation of the singular integrals around the chosen singular points is laborious. Ang (1986) has used both special Green's functions and the multidomain method to solve a multiple crack problem. Although this method avoids the discretisation and integration along the crack surfaces, the application is limited to the problem with the cracks specially distributed in an infinite domain and simple loading conditions. Recently Cruse and Novati (1992) employed the traction boundary integral equation of Weaver (1977) and Polch (1987) to deal with the multiple crack problem, but it is still limited to the analysis of an infinite body. Chen and Chen (1995) have developed a technique suitable for two dimensional finite geometries with multiple cracks. Based on the merits of the BEM, the displacement integral equation is derived for the outer boundary and the traction integral equation is established for only one of the crack edges/surfaces. Since the relative displacement on the crack edges/surfaces is taken as an independent variable in the formulation, the total number of degrees of freedom and computational efforts for multiple cracks are largely reduced. A virtual boundary connected to one of the crack edges to construct a closed integral path is employed for evaluating the hypersingular integral. The constant and quadratic isoparametric elements are considered to discretise the closed integrals paths/crack edges and outer boundary respectively. One of the examples studied by them is a finite plate with two inclined cracks (Fig. 11a). Because of symmetry, only one half of the plate is analysed. To compute the hypersingular integral, the crack surface is discretised with 18 constant elements. The outer boundary modelled by 20 quadratic elements is also shown (Fig. 11a). The variation of computed SIFs F_I ($F_I = K_I/\sigma\sqrt{\pi a}$) and F_{II} ($F_{II} = K_{II}/\sigma\sqrt{\pi a}$) with inclination angle θ are shown in Fig. 11b and c. Figure 11b shows that F_I at the crack tip A is always higher than that at the crack tip B due to the interaction between the two cracks. As the angle increases (and therefore two crack tips are closer), the difference increases. Because of geometric symmetry F_{II} at the crack tip A is lower than that at the crack tip B. These results are reported to be in good agreement with the finite element solutions (Chen and Chen, 1995).

Chen (1995, 1997) has proposed numerical solution of multiple crack problems by using hypersingular integral equation. He has suggested a new quadrature rule to numerically solve the hypersingular integral equation in the case of multiple cracks in a structure. Several examples, e.g. two cracks in series, interaction of a horizontal crack and an inclined crack, etc., are presented. According to Chen the proposed approach give very accurate results.

Recently, special BEs have been proposed by Mukhopadhyay et al. (1999d) which can simulate variable order singularities at the two ends of the element. The element, which models only the strain singularity at both the ends is termed as the Two End Strain Singularity (TESS) element. Similarly, the element which incorporates both the strain and traction singularities at both the ends is termed as the Two End Strain and Traction Singularity (TESTS) element. These elements have been developed by a manipulation of the element shape functions. Shape functions to give the singularity at the two ends can be written separately as per Eq. (18). To get a single set of shape functions to represent singularities simultaneously at the two ends a simple superposition is employed. A set of shape functions which can simulate variable order strain singularities at both the ends is as follows.

$$\begin{aligned} N_1 &= 2^{c-1}[-(r/l)^c + (r/l)^{c+1}] \\ &\quad + 2^{d-1}[-(1-r/l)^d + (1-r/l)^{d+1}] + 1 - (r/l) \\ N_2 &= 2^c[(r/l)^c - (r/l)^{c+1}] \\ &\quad + 2^d[(1-r/l)^d - (1-r/l)^{d+1}] \\ N_3 &= 2^{c-1}[-(r/l)^c + (r/l)^{c+1}] \\ &\quad + 2^{d-1}[-(1-r/l)^d + (1-r/l)^{d+1}] + (r/l) \end{aligned} \tag{20}$$

It may be noted that all the derivatives display r^{c-1} singularity at $r = 0$ and r^{d-1} at $r = l$. However, the traction does not display any singularity. Such an element is an example of the TESS element. To model both the strain and traction singularities, separate shape functions are needed to model the variable traction singularity. A set of such functions is given below.

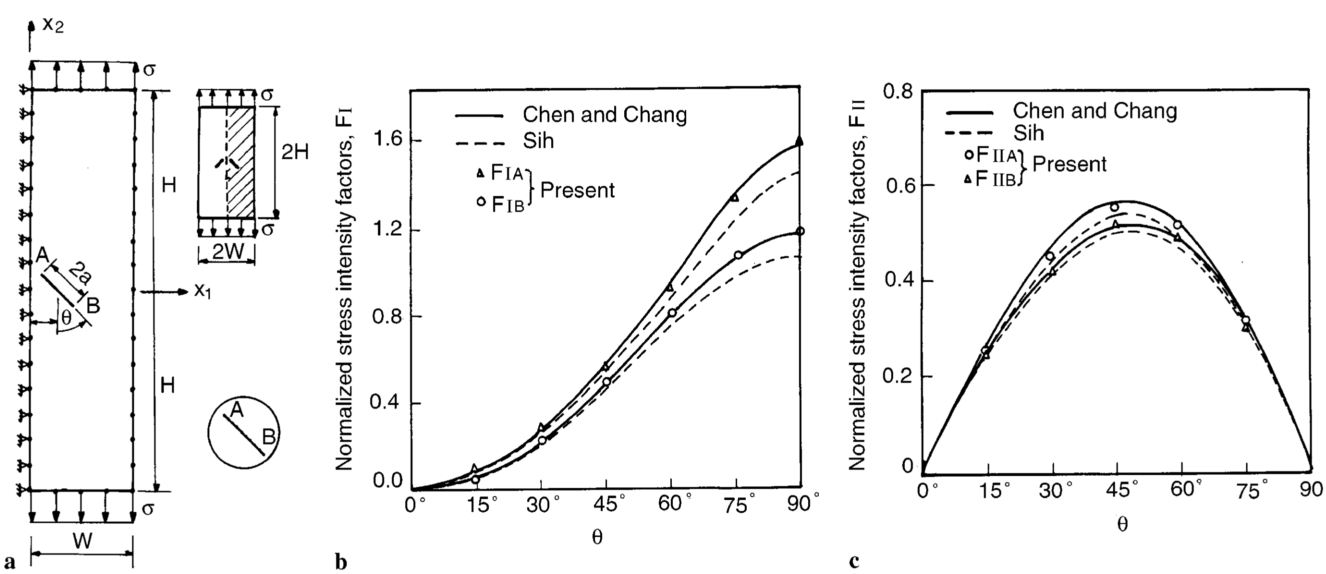


Fig. 11a–c. Plate with two inclined cracks. **a** Geometry and mesh. **b** Variation of normalised mode I SIF. **c** Variation of normalised mode II SIF (Chen and Chen, 1995)

$$
\begin{aligned}
M_1 &= 2^{c-1}[(r/l)^{c-1} - (r/l)^c] \\
&\quad + 2^d[-(1-r/l)^{d-1} + (1-r/l)^d] + 1 - (r/l) \\
M_2 &= 2^{c-1}[(r/l)^{c-1} - (r/l)^c] \\
&\quad + 2^{d-1}[(1-r/l)^{d-1} - (1-r/l)^d] \\
M_3 &= 2^c[-(r/l)^{c-1} + (r/l)^c] \\
&\quad + 2^{d-1}[(1-r/l)^{d-1} - (1-r/l)^d] + (r/l)
\end{aligned}
\tag{21}
$$

The simultaneous representation of displacement and traction by the shape functions (20) and (21) leads to a total modelling of the variable order singularities at both the end nodes. Mukhopadhyay et al. (1999d) have presented a procedure to evaluate the SERR based on the MCCI method in conjunction with these elements. The computed SIFs for a kinked crack emanating from a circular hole (Fig. 9b) are shown in Table 2. These special elements can cater for two dimensional applications. There is a need to examine how they can be extended to three dimensions to take care of both straight and curved crack fronts.

5 Summary

The BEM has been used extensively for the evaluation of the SIFs. Though the displacement method is a versatile and most widely used technique, the *J*-integral and MCCI based methods offer better accuracy. The *J*-integral method has been applied in both the subregion approach and the DBEM. In the MCCI based method, no additional computation of displacement and traction as required in the *J*-integral calculations, is necessary. The MCCI method can be applied to various types of loadings. The accuracy is good and the computed SIFs are less dependent on the crack tip element size than the displacement method. How this method can be applied in the DBEM requires some investigations.

It is clear, that modelling of total singularities, i.e. modelling of both strain and traction singularities simultaneously, help in improving the accuracy of the computed SIFs. The quarter point element can model the singularity partially. The quarter point traction singularity element can help to do both the strain and traction singularities. However this element can only handle the usual square root singularity. To solve problems of variable singularity or neighbouring singularities, a large number of conventional BEs or a few special singularity BEs can be employed. The latter provides an alternative attractive computationally. By a simple manipulation of the shape functions of a 3-noded quadratic element, variable order singularity can be obtained.

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