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FURTHER CONSIDERATIONS IN MODIFIED CRACK CLOSURE INTEGRAL BASED COMPUTATION OF STRESS INTENSITY FACTOR IN BEM

N. K. MUKHOPADHYAY and A. KAKODKAR†

Reactor Design & Development Group, Bhabha Atomic Research Centre, Mumbai 400085, India

S. K. MAITI

Mechanical Engineering Department, Indian Institute of Technology, Bombay, Mumbai 400076, India

Abstract—This paper deals with modified crack closure integral based computation of stress intensity factors in a boundary element method for problems with mechanical loading remote from the crack edges. The modified crack closure integral technique has been coupled with the local smoothing scheme to obtain simple relations for energy release rates for linear, quadratic and quarter point elements around the crack tip. Stress intensity factors calculated through the proposed formulation and the displacement method for a number of examples are compared, wherever possible, with data available in the literature. The results based on the proposed scheme are more accurate than those obtained by the displacement method. © 1998 Elsevier Science Ltd

Keywords—computation of SIF in BEM, crack closure integral, SIF, crack closure integral in BEM, accurate SIF computation.

NOMENCLATURE

a	crack length
c_n	coefficients of traction in CCI formulation
G_I, G_{II}	strain energy release rate in mode I, mode II
K	stress intensity factor
l	crack tip element length
p	internal pressure
r_1, r_2	internal and external radii
s_j	x-component of traction
t, tj	y-component of traction
u	x-component of displacement
v	y-component of displacement
W	crack closure work
x, y	cartesian coordinates
Y	SIF correction factor
μ	shear modulus
ν	Poisson's ratio
ξ	natural coordinate.

1. INTRODUCTION

The applications of boundary element method (BEM) for the evaluation of stress intensity factors (SIFs) have received a considerable attention [1–8]. The displacement method for the extraction of SIFs has been the basis to determine SIFs in the BEM. One of the important methods of determining SIF is based on crack closure integral technique (CCI). This technique has been used by several investigators to evaluate the SIFs in the FEM [9–14]. Recently the possibility of using the modified crack closure integral (MCCI) to evaluate SIFs in the BEM has been explored [15, 16]. The results are very encouraging. This paper deals with further considerations in this direction.

In the earlier paper [16], while obtaining the simultaneous equations involving nodal displacements and tractions, $\int u^* t \, d\Gamma$, where u^* is the fundamental solution for displacement and t is the specified traction, was evaluated over the portion OA (Fig. 1) immediately behind the crack

†Author to whom correspondence should be addressed.

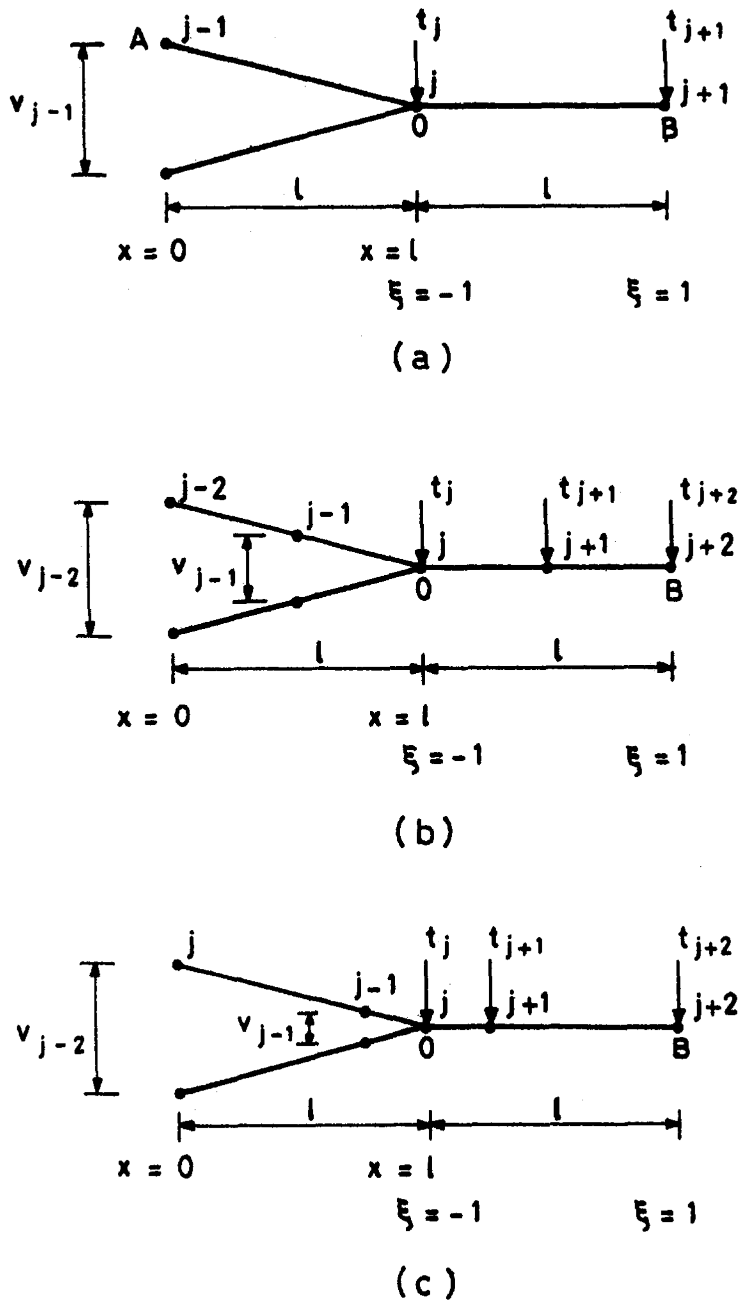

Fig. 1. (a) Linear element. (b) Quadratic element. (c) Quarter point element.

tip considering a distribution of traction arising out of traction t_0 at the crack tip node O. In the MCCI calculation too, the contribution to crack closure work due to the same traction variation over OA was included. For a traction free crack edge, while evaluating $\int u^* t \, d\Gamma$ over the portion OA, t can be taken as zero irrespective of the presence of t_0 at the crack tip. Furthermore, the crack closure work should be calculated with a similar presumption; it should receive contributions only from the element OB on the ligament side. The effect of such differences on the SIFs based on both the displacement and crack closure integral methods are presented. These are illustrated in the case of linear, quadratic and quarter point elements employed around the crack tip.

2. MATHEMATICAL FORMULATION

2.1. *Linear element*

For a typical discretisation (Fig. 1a), with crack tip at node j and the two adjacent nodes $j-1$ and $j+1$, the displacement variation over OA can be written in the form

$$v = v_{j-1}(1-\xi)/2 \tag{1}$$

where ξ is a natural coordinate and $\xi = 0$ at the middle of OA. A linear variation of the traction along OB can be represented in the same way:

$$t = 0.5(t_j + t_{j+1}) - 0.5(t_j - t_{j+1})\xi \tag{2}$$

where ξ is a natural coordinate. ξ is zero at the middle of OB.

While analysing a symmetric problem with discretisation of only one half or quarter of the domain, the unknown tractions over the ligament gives rise to a nodal traction at the crack tip node. The crack tip nodal traction can contribute routinely to a variation of traction over an element like OA (Fig. 1a). This variation contributes to the coefficients of the boundary element matrices associated with global displacements and tractions [16]. For traction-free crack edges, traction is zero over the span OA. This has to be imposed irrespective of a variation arising out of non-zero traction t_0 at the tip O. The contributions to the equation coefficients can therefore be neglected. Because of similar considerations the crack closure work needs to be calculated from the element OB ahead of the crack tip O. That is,

$$W = \frac{1}{2}\int_0^l vt\,\mathrm{d}x \tag{3}$$

On simplification

$$G_{\mathrm{I}} = v_{j-1}(c_1 t_j + c_2 t_{j+1})/12 \tag{4}$$

where $c_1 = 2$ and $c_2 = 1$.

A similar expression can be derived for the mode II strain energy release rate G_{II}, involving x components of tractions and displacements.

$$G_{\mathrm{II}} = u_{j-1}(c_1 s_j + c_2 s_{j+1})/12 \tag{5}$$

The SIF can then be calculated using the standard relations between G and K.

2.2. *Quadratic element*

In the case of quadratic elements around the crack tip (Fig. 1b) the displacement variation over OA is given by

$$v = v_{j-1} - 0.5v_{j-2}\xi + (0.5v_{j-2} - v_{j-1})\xi^2 \tag{6}$$

Similarly the traction variation, which is also quadratic, has the form

$$t = t_{j+1} + 0.5(t_{j+2} - t_j)\xi + [0.5(t_{j+2} + t_j) - t_{j+1}]\xi^2 \tag{7}$$

The crack closure work

$$W = \frac{1}{2}\int_0^l vt\,\mathrm{d}x$$

$$= [v_{j-1}(2t_j + 16t_{j+1} + 2t_{j+2}) + v_{j-2}(4t_j + 2t_{j+1} - t_{j+2})]/60 \tag{8}$$

The strain energy release rate

$$G_{\mathrm{I}} = [v_{j-1}(c_1 t_j + c_2 t_{j+1} + c_3 t_{j+2}) + v_{j-2}(c_4 t_j + c_5 t_{j+1} + c_6 t_{j+2})]/60 \tag{9}$$

where $c_1 = 2$, $c_2 = 16$, $c_3 = 2$, $c_4 = 4$, $c_5 = 2$, and $c_6 = -1$.

A similar expression for G_{II} can be obtained involving x-component of tractions and displacements:

$$G_{\mathrm{II}} = [u_{j-1}(c_1 s_j + c_2 s_{j+1} + c_3 s_{j+2}) + u_{j-2}(c_4 s_j + c_5 s_{j+1} + c_6 s_{j+2})]/60 \tag{10}$$

2.3. *Quarter point element*

In the case of quarter point elements (Fig. 1c) the displacement is assumed to vary as $\sqrt{x}$ along OA. That is

$$v = 2(v_{j-2} - 2v_{j-1})(1 - x/l) + (4v_{j-1} - v_{j-2})\sqrt{(1 - x/l)} \tag{11}$$

The traction too has a similar variation and can be represented in the form

$$t = t_j\{-0.5\xi(1-\xi)\} + t_{j+1}(1-\xi^2) + t_{j+2}\{0.5\xi(1+\xi)\} \tag{12}$$

where $1 + \xi = 2\sqrt{(x/l)}$.

The crack closure work

$$\begin{aligned} W &= \frac{1}{2}\int_0^l vt\,\mathrm{d}x \\ &= [v_{j-1}\{t_j(140-45\pi) + t_{j+1}(60\pi - 176) + t_{j+2}(56 - 15\pi)\} \\ &\quad + v_{j-2}\{t_j(11.25\pi - 34) + t_{j+1}(56 - 15\pi) + t_{j+2}(3.75\pi - 12)\}]/60 \end{aligned} \tag{13}$$

The strain energy release rate

$$G_{\mathrm{I}} = [v_{j-1}(c_1 t_j + c_2 t_{j+1} + c_3 t_{j+2}) + v_{j-2}(c_4 t_j + c_5 t_{j+1} + c_6 t_{j+2})]/60 \tag{14}$$

where

$$c_1 = (140 - 45\pi), \quad c_2 = (60\pi - 176), \quad c_3 = (56 - 15\pi)$$

$$c_4 = (11.25\pi - 34), \quad c_5 = (56 - 15\pi), \quad \text{and} \quad c_6 = (3.75\pi - 12)$$

A similar expression can be derived for G_{II}:

$$G_{\mathrm{II}} = [u_{j-1}(c_1 s_j + c_2 s_{j+1} + c_3 s_{j+2}) + u_{j-2}(c_4 s_j + c_5 s_{j+1} + c_6 s_{j+2})]/60 \tag{15}$$

3. CASE STUDIES

The problems of centre crack, edge crack and circular ring with radial cracks (Fig. 2) under mode I loading have been studied. The material is assumed to be isotropic with shear modulus $\mu = 10^5$ N/mm^2 and Poisson's ratio $\nu = 0.3$. All the examples have been studied using linear, quadratic and quarter point elements and assuming a plane strain condition. The numerical solutions are compared with the standard solutions available in the literature [17, 18]. All cases are analysed using single precision arithmetic on a PC486.

3.1. *Centre crack*

The crack length to width ratio a/w is considered in the range of 0.2 to 0.8 (Fig. 2a). This problem has a double symmetry and only one quarter of the plate is analysed. In the case of quadratic and quarter point elements the plate has been discretised using 22 elements and 44 nodes. The crack tip element size is $0.01a$. Subsequent elements, away from the crack tip, are $0.02a$, $0.04a$, $0.08a$, $0.15a$ etc. The same nodal arrangements have been employed for the case of linear element, the crack tip element size is $0.05a$ and the sizes of subsequent elements away from the crack tip vary accordingly. The total number of elements and nodes are 44. The SIF has been compared using both the displacement method and the proposed CCI method. In the displacement method the SIF is evaluated considering separately the displacement of the first and the second corner nodes behind the crack tip. The results in the form of SIF correction factor Y are compared with reference solutions, which are accurate within 1%, in Table 1. The

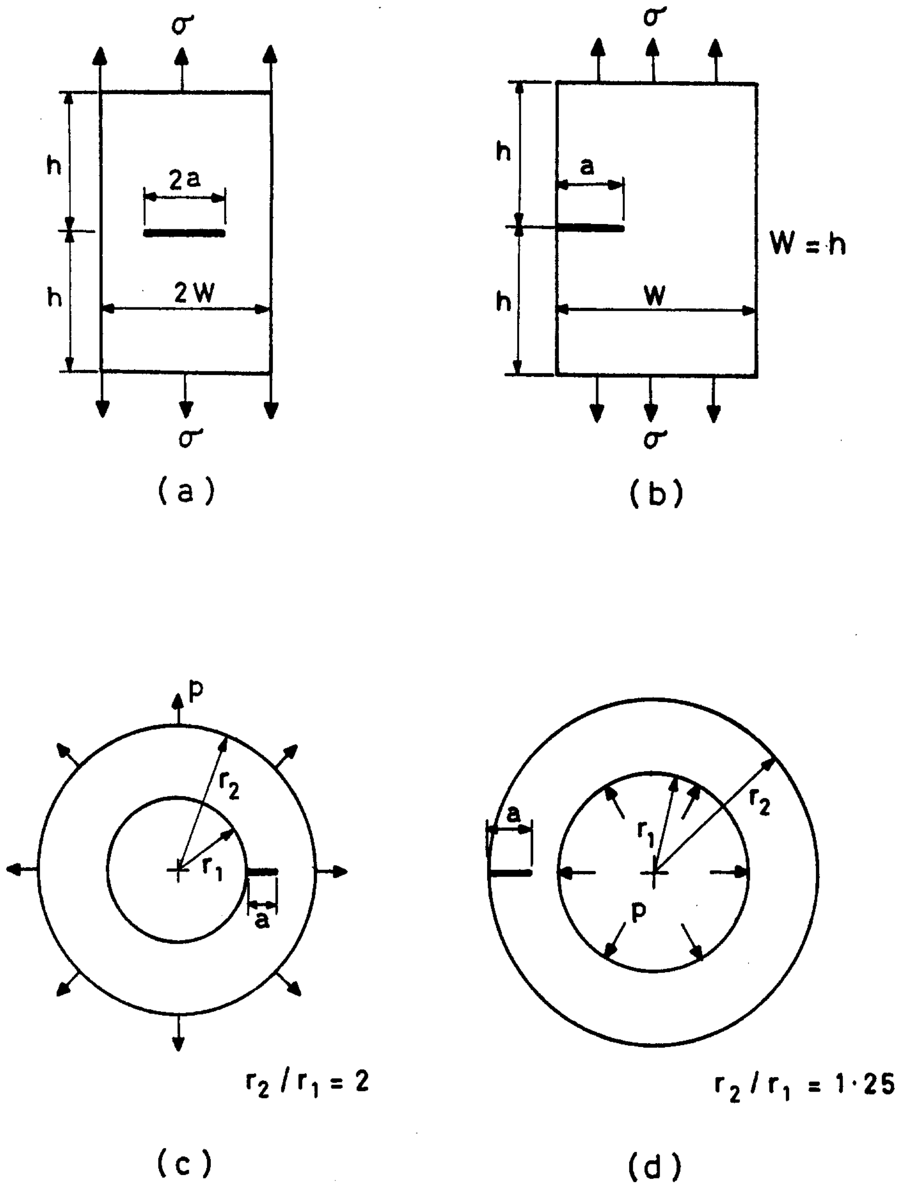

Fig. 2. Geometries considered for analysis. (a) Centre crack. (b) Edge crack. (c) Circular ring with radial inner edge crack. (d) Circular ring with radial outer edge crack.

effect of crack tip element size on the accuracy of the results has been studied considering $a/w = 0.5$. These results are plotted in Fig. 3.

The accuracy of the displacement method is dependent on where the displacements are compared. The error is reduced when the displacement is compared at the second corner node rather than the first one. The reduction is substantial in the case of linear elements where the maximum error reduces from 14% to about 9.5%. In the proposed CCI method this difference is within 8% when linear elements are employed. In the case of both quadratic and quarter point elements, the comparison of displacements at the second corner node is again preferable. However, in these cases, the CCI method gives rise to a substantial improvement in the accuracy. The error is less than 1.1% for quadratic elements.

3.2. *Edge crack*

The a/w ratio is considered in the range of 0.2–0.7 (Fig. 2b). Due to symmetry only one half of the problem has to be analysed. The earlier discretisations are again employed. The com-

Table 1. Comparison of SIF correction factor Y for centre crack

a/w	Reference solution	Displacement method: 1st Corner node Y	Displacement method: 1st Corner node % Error	Displacement method: 2nd Corner node Y	Displacement method: 2nd Corner node % Error	CCI Method Y	CCI Method % Error
				Linear element			
0.2	1.0254	0.9123	−11.027	0.9588	−6.499	0.9748	−4.934
0.3	1.0594	0.9402	−11.253	0.9880	−6.744	1.0046	−5.175
0.4	1.1118	0.9858	−11.329	1.0360	−6.820	1.0534	−5.254
0.5	1.1891	1.0524	−11.496	1.1059	−6.998	1.1245	−5.431
0.6	1.3043	1.1491	−11.902	1.2074	−7.428	1.2279	−5.861
0.7	1.4842	1.2928	−12.898	1.3594	−8.412	1.3807	−6.973
0.8	1.7989	1.5512	−13.771	1.6311	−9.330	1.6567	−7.906
				Quadratic element			
0.2	1.0254	0.9679	−5.610	0.9875	−3.699	1.0246	−0.075
0.3	1.0594	0.9979	−5.633	1.0199	−3.733	1.0584	−0.095
0.4	1.1118	1.0499	−5.569	1.0710	−3.670	1.1115	−0.027
0.5	1.1891	1.1221	−5.636	1.1446	−3.744	1.1880	−0.096
0.6	1.3043	1.2258	−6.021	1.2502	−4.148	1.2978	−0.498
0.7	1.4842	1.3895	−6.383	1.4166	−4.554	1.4714	−0.865
0.8	1.7989	1.6799	−6.613	1.7115	−4.857	1.7795	−1.078
				Quarter point element			
0.2	1.0254	1.0015	−2.329	1.0018	−2.299	0.9942	−3.042
0.3	1.0594	1.0340	−2.400	1.0342	−2.377	1.0265	−3.101
0.4	1.1118	1.0866	−2.264	1.0869	−2.244	1.0788	−2.967
0.5	1.1891	1.1615	−2.319	1.1617	−2.304	1.1532	−3.018
0.6	1.3043	1.2694	−2.676	1.2694	−2.675	1.2604	−3.367
0.7	1.4842	1.4393	−3.028	1.4388	−3.062	1.4293	−3.701
0.8	1.7989	1.7414	−3.195	1.7396	−3.298	1.7298	−3.839

puted SIF correction factor Y based on the displacement method and the CCI technique have been compared in Table 2. The effect of crack tip element size for $a/w = 0.5$ has been studied. The corresponding results are presented in Fig. 4.

In this case again a comparison of displacement at the second corner node is preferable. For a comparison of displacement at the first corner node, the error is around 20% for the linear element, 5% for the quadratic element and less than 1.5% for the quarter point element. For a comparison of displacement at the second corner node the error is within 16, 2.5 and 1.5% for the linear, quadratic and quarter point elements, respectively. In the CCI method the error is within 15.5% in the case of linear element. The error reduces drastically when the quadratic or quarter point elements are employed. The maximum error is 2.2% for the range of $a/w = 0.2$–0.7 for both the quadratic and quarter point elements.

3.3. *Circular ring with radial crack*

The first example of circular ring deals with radial inner edge crack under external uniform tension (Fig. 2c). The parameter $a/(r_2-r_1)$ is considered in the range of 0.2–0.8. The number of elements are 26 and 52, respectively, when quadratic and quarter point elements are used. For linear element the same discretisation has been adapted and the number of elements and nodes are 52. The elements near the crack tip are taken the same way as in the case of the centre crack example. The results are compared with reference solutions which are accurate within 1%, in Table 3. The trend is similar to the earlier observations on the centre crack and edge crack. The error is found to be less than 1.22% and 4% for the range of $a/(r_2-r_1) = 0.2$–0.8 when quadratic and quarter point elements are employed, respectively, in conjunction with the proposed method.

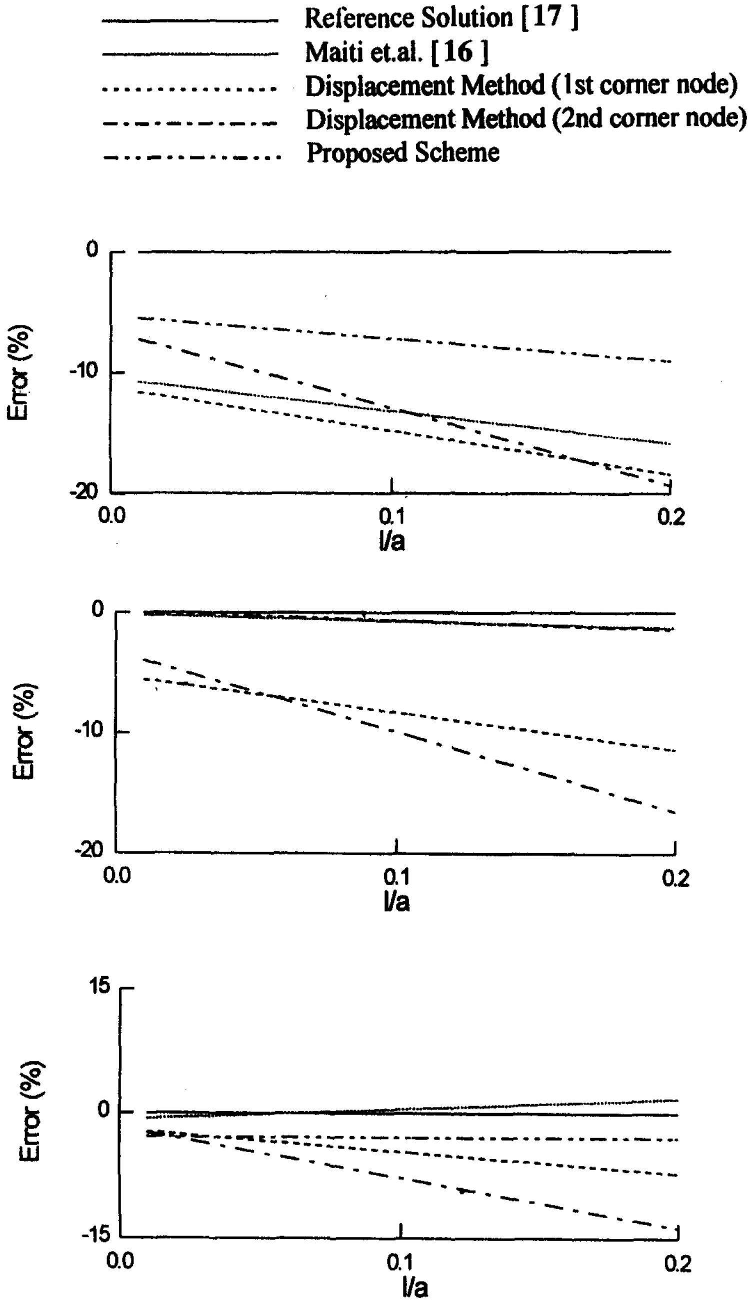

Fig. 3. Effect of crack tip element size on error in the case of the centre crack. (a) Linear element. (b) Quadratic element. (c) Quarter point element.

The last case deals with a circular ring with radial outer edge crack under uniform internal pressure (Fig. 2 d). The SIF correction factor Y for this case, where no comparison has been possible, is presented in Table 4. The results show that, for a/w, for example, equal to 0.3, the difference with Rooke and Cartwright [18] is less than 6, 1.5 and 4% based on the linear element, quadratic element and quarter point element, respectively, in conjunction with the proposed scheme.

The difference between the present results and those reported earlier in ref. [16] is illustrated in Table 5. The effect of crack tip element size on the accuracy of computation of SIF correction factor Y is again compared with earlier data [16] in Figs 3 and 4.

Table 2. Comparison for SIF correction factor Y for edge crack

		SIF Correction factor Y					
		Computed by					
		Displacement method				CCI Method	
	Reference	1st Corner node		2nd Corner node			
a/w	solution	Y	% Error	Y	% Error	Y	% Error
				Linear element			
0.2	1.3736	1.2361	−10.012	1.2999	−5.364	1.3200	−3.904
0.3	1.6629	1.4939	−10.161	1.5717	−5.487	1.5949	−4.091
0.4	2.1066	1.8902	−10.272	1.9898	−5.544	2.0170	−4.255
0.5	2.8297	2.5048	−11.481	2.6394	−6.727	2.6707	−5.618
0.6	4.0299	3.5078	−12.956	3.7019	−8.138	3.7355	−7.306
0.7	6.3610	5.0396	−20.773	5.3561	−15.798	5.3802	−15.419
				Quadratic element			
0.2	1.3736	1.3088	−4.716	1.3393	−2.497	1.3836	0.731
0.3	1.6629	1.5912	−4.313	1.6306	−1.940	1.6810	1.088
0.4	2.1066	2.0259	−3.833	2.0813	−1.202	2.1377	1.478
0.5	2.8297	2.7109	−4.199	2.7959	−1.193	2.8553	0.904
0.6	4.0299	3.8718	−3.923	4.0180	−0.294	4.0660	0.895
0.7	6.3610	6.0693	−4.585	6.2118	−2.345	6.3845	0.369
				Quarter point element			
0.2	1.3736	1.3551	−1.348	1.3596	−1.021	1.3433	−2.202
0.3	1.6629	1.6482	−0.886	1.6561	−0.412	1.6328	−1.811
0.4	2.1066	2.0994	−0.343	2.1146	0.382	2.0772	−1.393
0.5	2.8297	2.8114	−0.646	2.8429	0.466	2.7766	−1.878
0.6	4.0299	4.0198	−0.250	4.0899	1.488	3.9579	−1.786
0.7	6.3610	6.2973	−1.001	6.3577	−0.052	6.2226	−2.176

4. DISCUSSION

The accuracy of the displacement method is always dependent on where the displacements are compared. The accuracy of SIF improves substantially in the case of all the three crack tip elements by comparing the displacement at the second corner node rather than the first one.

The comparisons presented in Table 5 indicate no significant change in the accuracy of the computed SIF correction factor Y for the quadratic element. The maximum error observed earlier[16] is 1.173, 1.180 and 1.468% for the case of centre crack, edge crack and circular ring with radial inner edge crack, respectively. The present results show a maximum error of 1.078, 1.478 and 1.218%, respectively, for the three cases. Improvement in the accuracy of computed Y is more significant in the case of linear element. The present error is 8, 15.5 and 9.2% as compared to the earlier values, 11.4, 15.8 and 13.2%, respectively, in the above three examples. In the case of quarter point element, the present maximum error is 4% as compared to the maximum of 2% observed earlier. On the whole, the present method of calculation is recommended when crack tip singularity elements are not employed.

The accuracy of correction factor Y based on the displacement method shows more improvement in the case of quarter point elements than the quadratic elements. However, when the CCI method is employed the same trend is not observed. This may be due to the fact that, while using the singularity elements, the strain singularity is ensured but not the traction singularity.

The effect of crack tip element size on the accuracy of SIF shows (Figs 3 and 4) that the displacement method is very sensitive to the crack tip element size. The accuracy reduces, as expected, with an increase in the element size. For example, the minimum difference in both the cases (Figs 3 and 4) is about 11.5% for a crack tip element size of $0.01a$ when linear element is employed and displacement is compared at the first corner node. This increases to 19 and 16%, respectively, for a crack tip element size of $0.2a$.

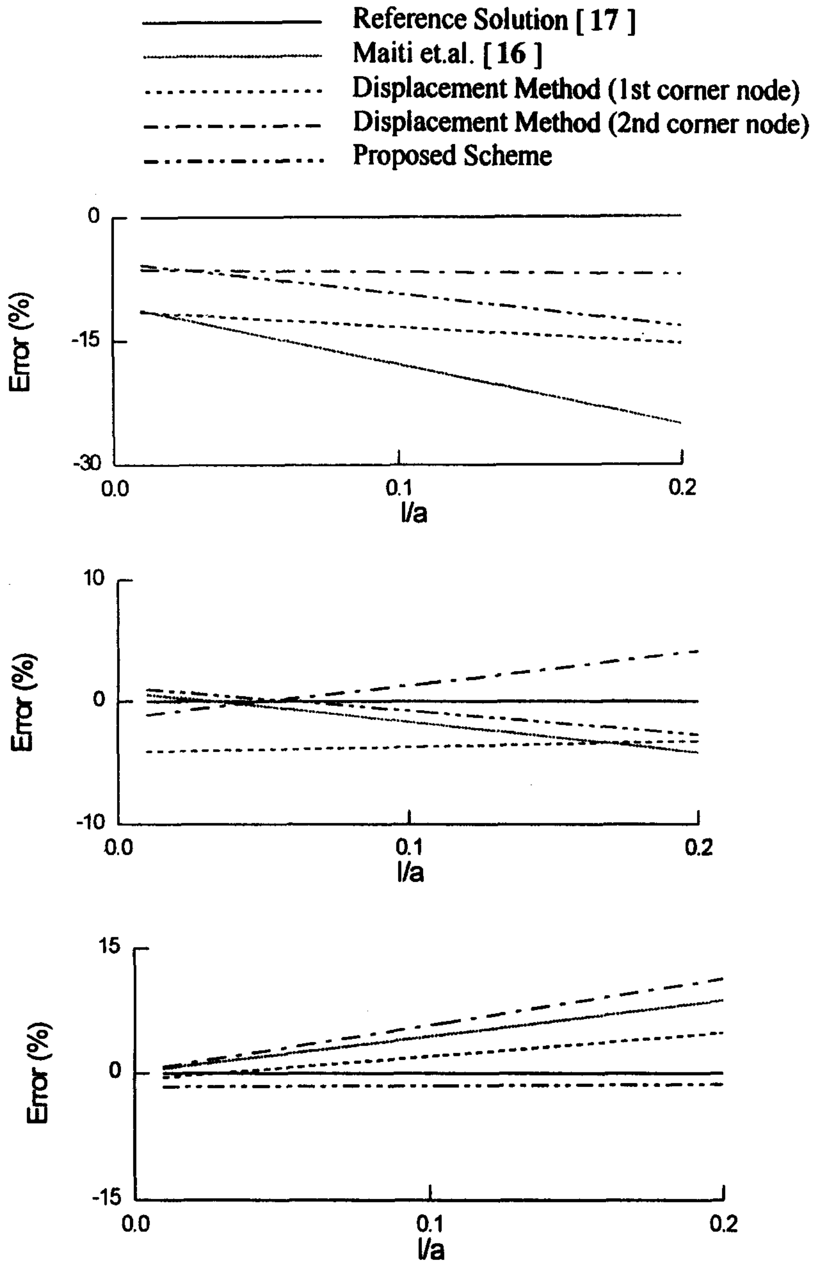

Fig. 4. Effect of crack tip element size on error in the case of the edge crack. (a) Linear element. (b) Quadratic element. (c) Quarter point element.

The proposed CCI method is less sensitive to the mesh refinement. In the case of the linear elements the difference with the reference solution increases steadily with the element size. The minimum difference observed is about 5.5% for a crack tip element size of $0.01a$ for both the cases. The difference is around 9 and 13% for the centre and edge crack problems, respectively, for a crack tip element size of $0.2a$. In the case of quadratic elements, for a crack tip element size up to $0.2a$, the difference is around 1.4% for the centre crack and less than 3% for the edge crack. In the case of the quarter point elements the error is within 3 and 2%, respectively, for an element size up to $0.2a$. These results, when compared with results reported earlier[16], show no significant change for the quadratic elements (Figs 3 and 4).

Table 3. Comparison of SIF correction factor Y for circular ring with radial inner edge crack under external tension

	SIF Correction factor Y							
		Computed by						
		Displacement method				CCI Method		
	Reference	1st Corner node		2nd Corner node				
$a/(r_2-r_1)$	solution	Y	% Error	Y	% Error	Y	% Error
			Linear element				
0.2	2.7760	2.4510	−11.707	2.5791	−7.093	2.6161	−5.760
0.3	2.8672	2.5010	−12.771	2.6313	−8.226	2.6698	−6.883
0.4	2.9887	2.5901	−13.337	2.7250	−8.823	2.7651	−7.482
0.5	3.1360	2.6809	−14.514	2.8213	−10.035	2.8612	−8.763
0.6	3.3152	2.8324	−14.564	2.9812	−10.074	3.0225	−8.828
0.7	3.5541	3.0268	−14.837	3.1871	−10.327	3.2291	−9.145
0.8	3.9125	3.3527	−14.309	3.5319	−9.729	3.5753	−8.618
			Quadratic element				
0.2	2.7760	2.6203	−5.608	2.6854	−3.264	2.7677	−0.299
0.3	2.8672	2.7013	−5.786	2.7678	−3.466	2.8536	−0.473
0.4	2.9887	2.8199	−5.647	2.8899	−3.306	2.9789	−0.329
0.5	3.1360	2.9469	−6.030	3.0238	−3.578	3.1111	−0.794
0.6	3.3152	3.1192	−5.913	3.2040	−3.355	3.2914	−0.717
0.7	3.5541	3.3299	−6.308	3.4279	−3.550	3.5108	−1.218
0.8	3.9125	3.6747	−6.077	3.7877	−3.190	3.8709	−1.065
			Quarter point element				
0.2	2.7760	2.7108	−2.348	2.7241	−1.870	2.6848	−3.284
0.3	2.8672	2.7955	−2.502	2.8084	−2.050	2.7692	−3.417
0.4	2.9887	2.9191	−2.329	2.9331	−1.859	2.8916	−3.248
0.5	3.1360	3.0507	−2.720	3.0692	−2.132	3.0200	−3.699
0.6	3.3152	3.2299	−2.572	3.2529	−1.878	3.1961	−3.592
0.7	3.5541	3.4494	−2.946	3.4814	−2.046	3.4101	−4.052
0.8	3.9125	3.8090	−2.646	3.8491	−1.622	3.7620	−3.846

5. CONCLUSIONS

A method of boundary element based evaluation of the SIFs has been proposed. Four examples have been presented to illustrate the accuracy of this scheme. The case studies show that the modified CCI technique in conjunction with the local smoothing scheme helps in improving the accuracy of computation of the SIFs. The results based on the modified CCI formulation is better than the displacement method in the case of linear elements. Further improvements are observed by using the quadratic elements. The crack tip element size of up to $0.1a$ is recommended for good accuracy. A utilisation of special crack tip elements may not lead to any extra advantage.

Table 4. SIF correction factor Y for circular ring with radial outer edge crack using CCI method

	SIF Correction factor Y computed by		
	Linear element	Quadratic element	Quarter point element
$a/(r_2-r_1)$	Y	Y	Y
0.1	1.1205	1.1690	1.1430
0.2	1.2487	1.3028	1.2738
0.3	1.4538	1.5169	1.4830
0.4	1.7408	1.8204	1.7797
0.5	2.1156	2.2291	2.1815
0.6	2.5623	2.7174	2.6592

Table 5. Comparison of SIF correction factor Y using CCI method

	SIF Correction factor Y						
		Linear element		Quadratic element		Quarter point element	
		% Error		% Error		% Error	
a/w	Reference solution	Ref.[16]	Present	Ref.[16]	Present	Ref.[16]	Present
				Centre crack			
0.2	1.0254	−10.002	−4.934	−0.270	−0.075	−0.944	−3.042
0.3	1.0594	−10.372	−5.175	−0.204	−0.095	−0.735	−3.101
0.4	1.1118	−10.510	−5.254	−0.217	−0.027	−0.857	−2.967
0.5	1.1891	−10.670	−5.431	−0.289	−0.096	−0.898	−3.018
0.6	1.3043	−9.103	−5.861	−0.760	−0.498	−1.201	−3.367
0.7	1.4842	−11.125	−6.973	−0.871	−0.865	−1.263	−3.701
0.8	1.7989	−11.388	−7.906	−1.173	−1.078	−1.458	−3.839
				Edge crack			
0.2	1.3736	−9.207	−3.904	0.418	0.731	−0.075	−2.202
0.3	1.6629	−9.309	−4.091	0.828	1.088	0.396	−1.811
0.4	2.1066	−9.463	−4.255	1.180	1.478	0.848	−1.393
0.5	2.8297	−10.743	−5.618	0.627	0.904	0.482	−1.878
0.6	4.0299	−12.675	−7.306	0.543	0.895	0.703	−1.786
0.7	6.3610	−15.776	−15.419	−0.026	0.369	0.093	−2.176
				Circular ring with radial inner edge crack			
0.2	2.7760	−8.958	−5.760	−0.503	−0.299	−1.301	−3.284
0.3	2.8672	−10.301	−6.883	−0.682	−0.473	−1.388	−3.417
0.4	2.9887	−11.602	−7.482	−0.539	−0.329	−1.160	−3.148
0.5	3.1360	−12.445	−8.763	−1.017	−0.794	−1.629	−3.699
0.6	3.3152	−12.662	−8.828	−0.950	−0.717	−1.493	−3.592
0.7	3.5541	−13.174	−9.145	−1.468	−1.218	−1.929	−4.052
0.8	3.9125	−12.796	−8.618	−1.331	−1.065	−1.625	−3.846

REFERENCES

1. Cruse, T. A. and Van Buren, W., *International Journal of Fracture Mechanics*, 1971, **7**, 1–16.
2. Cruse, T. A., *Applied Mathematical Modelling*, 1978, **2**, 287–293.
3. Tan, C. L. and Fenner, R. T., *Journal of Strain Analysis*, 1978, **13**, 231–237.
4. Blandford, G. E., Ingraffea, A. R. and Liggett, J. A., *International Journal of Numerical Methods in Engineering*, 1981, **17**, 387–404.
5. Liu, S. B. and Tan, C. L., *Engineering and Fracture Mechanics*, 1992, **42**, 273–288.
6. Ang, H. E., Torrance, J. E. and Tan, C. L., *Engineering and Fracture Mechanics*, 1996, **54**, 601–615.
7. Gerstle, W. H., Ingraffea, A. R. and Perucchio, R., *International Journal of Fatigue*, 1988, **10**, 189–192.
8. Rigby, R. H. and Aliabadi, M. H., Boundary element analysis of three-dimensional crack problems. In *Computational Methods in Fracture Mechanics*, ed. Aliabadi, M. H., Nisatani, H. and Cartwright, D. J., Computational Mechanics Publications. Southampton, 1992, pp. 91–105.
9. Rybicki, E. F. and Kanninen, M. F., *Engineering and Fracture Mechanics*, 1977, **9**, 931–938.
10. Krishnamurthy, T., Ramamurthy, T. S., Vijayakumar, K. and Duttaguru, B., Modified crack closure integral method for higher order finite elements. In *Proceedings of the International Conference on Finite Elements in Computational Mechanics*, ed. Kant, T. Pergamon Press, Oxford, 1995, pp. 891–900.
11. Ramamurthy, T. S., Krishnamurthy, T., Badri, Narayan K., Vijaykumar, K. and Duttaguru, B., *Mechanical Research Communications*, 1986, **13**, 179–186.
12. Sethuraman, R. and Maiti, S. K., *Engineering and Fracture Mechanics*, 1988, **30**, 227–231.
13. Maiti, S. K., *International Journal of Fracture*, 1990, **43**, 161–174.
14. Maiti, S. K., *Engineering and Fracture Mechanics*, 1992, **41**, 339–348.
15. Farris, T. N. and Liu, M., *International Journal of Fracture*, 1993, **60**, 33–47.
16. Maiti, S. K., Mukhopadhyay, N. K. and Kakodkar, A., *Computational Mechanics*, 1997, **19**, 203–210.
17. Murakami, Y., *Stress Intensity Factors Handbook*. Pergamon Press, New York, 1987.
18. Rooke, D. P. and Cartwright, D. J., *Compendium of Stress Intensity Factors*. HMSO, London, 1976.

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